6.1 DETERMINANTS

RECALL THAT FOR $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2$ WE DEFINED

$$\det(A) = ad - bc,$$

AND WE OBSERVED THAT A is INVERTIBLE IF AND ONLY IF $\det(A) \neq 0$. OUR GOAL is NOW TO EXTEND THIS DEFINITION TO SQUARE MATRICES OF ANY SIZE. i.e. WE SEEK A FUNCTION

$$\det : M_n \to \mathbb{R}$$

SUCH THAT $A \in M_n$ is INVERTIBLE IF AND ONLY IF $\det(A) \neq 0$.

<u>DEFN.</u>
LET

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & & & \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix} \in M_n$$

AND WRITE A_{ij} FOR THE $(n-1)\times(n-1)$ MATRIX OBTAINED FROM A BY DELETING ROW i AND COLUMN j

$$A_{ij} = \begin{pmatrix} a_{11} & a_{12} & \cdots & \cancel{a_{1j}} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cancel{a_{2j}} & \cdots & a_{2n} \\ \vdots & \vdots & & & & \\ \cancel{a_{i1}} & \cancel{a_{i2}} & & \cancel{a_{ij}} & \cdots & \cancel{a_{in}} \\ \vdots & \vdots & & & & \\ a_{n1} & a_{n2} & \cdots & \cancel{a_{nj}} & \cdots & a_{nn} \end{pmatrix} \in M_{n-1}$$

DEFINE

$$\det(A) = a_{11}\det A_{11} - a_{21}\det A_{21} + \cdots + (-1)^{n+1} a_{n1}\det A_{n1}$$
$$= \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det(A_{i,1})$$

OBSERVE THAT THIS DEFINITION IS RECURSIVE SINCE IN ORDER TO COMPUTE AN $n \times n$ DETERMINANT, YOU MUST FIRST COMPUTE n $(n-1)\times(n-1)$ DETERMINANTS. THE RECURSION 'BOTTOMS OUT' AT $n=1$, THUS

$\underline{n=1}$ $\det(a) = a$

$\underline{n=2}$ $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

$\underline{n=3}$ $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a(ei-fh) - d(bi-ch) + g(bf-ce)$

$\underline{n=4}$

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = a_{11}\det\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} - a_{21}\det\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$
$$+ a_{31}\det\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} - a_{41}\det\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

Ex.

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \\ -1 & 3 & -2 \end{pmatrix} = 1\cdot(0-15) - 4(-4-9) + (-1)(10-0)$$

$$= -15 + 52 - 10 = \boxed{27}.$$

THEOREM

WE CAN COMPUTE $\det(A)$ BY EXPANDING ALONG ANY ROW OR COLUMN, i.e.

FOR ANY FIXED i $(1 \le i \le n)$:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

AND FOR ANY FIXED j $(1 \le j \le n)$:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

THESE SUMMATIONS ARE CALLED <u>LAPLACE EXPANSIONS</u>.
IT IS USEFUL TO REMEMBER THE SIGN MATRIX

$$\left((-1)^{i+j}\right) = \begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Ex.
SAME EXAMPLE, ALONG COLUMN 3:

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \\ -1 & 3 & -2 \end{pmatrix} = 3(12-0) - 5(3+2) + (-2)(0-8)$$
$$= 36 - 25 + 16 = \boxed{27}.$$

ALONG ROW 2:

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \\ -1 & 3 & -2 \end{pmatrix} = -4(-4-9) + 0(\quad) - 5(3+2)$$
$$= 52 - 25 = \boxed{27}.$$

Ex.
IT'S USEFUL TO PICK A ROW OR COLUMN WITH AS MANY ZEROS AS POSSIBLE.

$$\det\begin{pmatrix} 1 & 4 & 3 & -1 \\ 2 & 0 & 2 & 1 \\ 0 & 6 & 0 & 4 \\ -2 & -3 & 4 & 0 \end{pmatrix} = -6\det\begin{pmatrix} 1 & 3 & -1 \\ 2 & 2 & 1 \\ -2 & 4 & 0 \end{pmatrix} - 4\det\begin{pmatrix} 1 & 4 & 3 \\ 2 & 0 & 2 \\ -2 & -3 & 4 \end{pmatrix}$$

$$= -6\Big[(-1)(8+4) - 1\cdot(4+6)\Big] - 4\Big[-2(16+9) - 2(-3+8)\Big]$$

$$= (-6)(-22) - 4(-60) = \boxed{372}$$

PARTIONED MATRICES

SUPPOSE $A, B, C \in M_2$ AND CONSIDER

$$\begin{pmatrix} A & B \\ O & C \end{pmatrix} \in M_4$$

WHERE $O \in M_2$ IS THE ZERO MATRIX.
WE HAVE

$$\det \begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ 0 & 0 & c_{11} & c_{12} \\ 0 & 0 & c_{21} & c_{22} \end{pmatrix}$$

$$= a_{11} \det \begin{pmatrix} a_{22} & b_{21} & b_{22} \\ 0 & c_{11} & c_{12} \\ 0 & c_{21} & c_{22} \end{pmatrix} - a_{21} \det \begin{pmatrix} a_{12} & b_{11} & b_{12} \\ 0 & c_{11} & c_{12} \\ 0 & c_{21} & c_{22} \end{pmatrix}$$

$$= a_{11}a_{22} \det(C) - a_{21}a_{12} \det(C)$$

$$= (a_{11}a_{22} - a_{21}a_{12}) \det(C)$$

THUS

$$\det \begin{pmatrix} A & B \\ O & C \end{pmatrix} = \det(A) \cdot \det(C)$$

MORE GENERALLY, WE HAVE

THEOREM

IF A, C ARE SQUARE MATRICES (NOT NECESSARILY THE SAME SIZE) THEN

$$\det\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det(A)\cdot\det(C),$$

ALSO

$$\det\begin{pmatrix} A & 0 \\ B & C \end{pmatrix} = \det(A)\cdot\det(C),$$

FOR ANY (APPROPRIATELY SIZED) MATRIX B.

NOTE HOWEVER THAT IN GENERAL

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} \neq \det(A)\det(D) - \det(B)\det(C)$$

Ex.

$$\det\begin{pmatrix} 5 & 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix} = \det\begin{pmatrix} 5 & 4 \\ 6 & 7 \end{pmatrix}\cdot\det\begin{pmatrix} 5 & 6 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= (35-24)\cdot 5(1-0)$$

$$= 11\cdot 5 = \boxed{55}.$$

THEOREM

THE DETERMINANT OF AN UPPER (OR LOWER) TRIANGULAR MATRIX IS THE PRODUCT OF ITS DIAGONAL ENTRIES

$$\det\begin{pmatrix} a_{11} & * & * & \cdots & * \\ 0 & a_{22} & * & \cdots & * \\ 0 & 0 & a_{33} & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} = a_{11}a_{22}a_{33}\cdots a_{nn}$$

HW (6.1): 6, 8, 10, 14, 16, 18, 20, 28, 30, 34, 38, 44