4.3 MATRIX OF A LINEAR TRANSFORMATION

LET V BE A LINEAR SPACE WITH BASIS

$$\mathfrak{B} = \{ f_1, \ldots, f_n \}$$

AND LET $T: V \to V$ BE LINEAR. LET $F: \mathbb{R}^n \to \mathbb{R}^n$ BE THE MAPPING

$$F\left([f]_{\mathfrak{B}} \right) = \left[T(f) \right]_{\mathfrak{B}}$$

FOR ANY $f \in V$. i.e. F IS THE MAPPING WHICH MAKES THE FOLLOWING DIAGRAM COMMUTE

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d] & V \arrow[d] \\ \mathbb{R}^n \arrow[r, "F"'] & \mathbb{R}^n \end{tikzcd}$$

: $\mathfrak{B}$-COORDINATE TRANSFORMATION

<u>EXERCISE</u>: SHOW THAT F IS LINEAR.

LET B BE THE MATRIX OF F, i.e. B IS DEFINED BY THE RELATION

$$B\,[f]_{\mathfrak{B}} = \left[T(f) \right]_{\mathfrak{B}}$$

FOR ALL $f \in V$. WE CALL B THE <u>MATRIX OF T WITH RESPECT TO $\mathfrak{B}$</u>

TO DETERMINE THE COLUMNS OF B WE COMPUTE $B\vec{e}_j$, WHERE $\vec{e}_j$ IS THE j^{TH} STANDARD BASIS VECTOR IN $\mathbb{R}^n$ $(1 \le j \le n)$.

OBSERVE $\vec{e}_j = [f_j]_\beta$ SINCE

$$f_j = 0 \cdot f_1 + \cdots + 0 \cdot f_{j-1} + 1 \cdot f_j + 0 \cdot f_{j+1} + \cdots + 0 \cdot f_n$$

THUS

$$B\vec{e}_j = B[f_j]_\beta = [T(f_j)]_\beta$$

WHENCE

$$B = \Big[\, [T(f_1)]_\beta \cdots [T(f_n)]_\beta \,\Big] \qquad (n \times n).$$

EX.

DEFINE $T : M_2 \to M_2$ BY

$$T(A) = A\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} A$$

FOR ALL $A \in M_2$. OBSERVE THAT T IS LINEAR (EXERCISE), AND $\ker(T)$ CONSISTS OF ALL $A \in M_2$ WHICH COMMUTE WITH THE MATRIX

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}.$$

CONSIDER THE BASIS

$$\mathcal{B} = \left\{ \underset{E_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{E_2}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{E_3}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \underset{E_4}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \right\}$$

THIS BASIS DEFINES AN ISOMORPHISM FROM M_2 TO $\mathbb{R}^4$ GIVEN BY

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

THIS ISOMORPHISM IS NONE OTHER THAN THE $\mathcal{B}$-COORDINATE TRANSFORMATION.

$$T(E_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2E_2$$

$$T(E_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2E_2$$

$$T(E_3) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ -2 & 2 \end{pmatrix}$$

$$= (-2)E_1 + (-2)E_3 + 2E_4$$

$$T(E_4) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} = (-2)E_2$$

THUS THE MATRIX OF T WITH RESPECT TO $\mathcal{B}$ IS

$$B = \begin{pmatrix} 0 & 0 & -2 & 0 \\ 2 & 2 & 0 & -2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

OBSERVE

$$\mathrm{RREF}(B) = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

WE HAVE THE FOLLOWING BASES

$\underline{\mathrm{im}(B)}$

$$\left\{ \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -2 \\ -2 \end{pmatrix} \right\}$$

$\underline{\mathrm{im}(T)}$

$$\left\{ \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ -2 & -2 \end{pmatrix} \right\}$$

$\underline{\mathrm{KER}(B)}$

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\underline{\mathrm{KER}(T)}$

$$\left\{ \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

THEREFORE $\mathrm{rank}(T) = 2$ AND $\mathrm{nullity}(T) = 2$.

Ex.

LET $U_2 \subseteq M_2$ BE THE SET OF ALL UPPER TRIANGULAR MATRICES IN M_2, i.e.

$$U_2 = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$$

EXERCISE: SHOW THAT U_2 IS A SUBSPACE OF M_2 WITH BASIS

$$\left\{ \underset{E_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{E_2}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{E_4}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \right\}$$

EXERCISE: SHOW THAT U_2 IS CLOSED UNDER MATRIX MULTIPLICATION, i.e.

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

THUS $T: M_2 \to M_2$ FROM THE PREVIOUS EXAMPLE DEFINES A LINEAR MAP $U_2 \to U_2$ CALLED THE <u>RESTRICTION OF T TO U_2</u>, AND DENOTED

$$T\big|_{U_2} : U_2 \to U_2$$

i.e.

$$T\big|_{U_2}(A) = A\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} A$$

FOR ALL $A \in U_2$. OUR GOAL IN THIS EXAMPLE IS TO FIND THE MATRIX OF $T|_{U_2}$ WITH RESPECT TO $\{E_1, E_2, E_4\}$.

RECALL FROM THE EARLIER EXAMPLE

$$T\big|_{U_2}(E_1) = T(E_1) = 2E_2$$

$$T\big|_{U_2}(E_2) = T(E_2) = 2E_2$$

$$T\big|_{U_2}(E_4) = T(E_4) = -2E_2$$

THUS THE MATRIX OF $T|_{U_2}$ WITH RESPECT TO $\{E_1, E_2, E_3\}$ IS

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

OBSERVE

$$\mathrm{RREF}(C) = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

THEREFORE WE HAVE BASES

$\underline{\mathrm{Im}(C)}$

$$\left\{ \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\}$$

$\underline{\mathrm{Im}(T|U_2)}$

$$\left\{ \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \right\}$$

$\underline{\mathrm{KER}(C)}$

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

KER(T

$$\left\{ \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

WHENCE

$$\mathrm{rank}(T|U_2) = 1 \quad \text{AND} \quad \mathrm{nullity}(T|U_2) = 2$$

NOW LET V BE A FINITE DIMENSIONAL LINEAR SPACE, AND SUPPOSE

$$\mathcal{B} = \{ f_1, \dots, f_n \}$$

AND

$$\mathcal{a} = \{ g_1, \dots, g_n \}$$

ARE TWO BASES OF V.

<u>EXERCISE</u>

SHOW THAT THE MAPPING $\mathbb{R}^n \to \mathbb{R}^n$ WHICH TAKES $[f]_{\mathcal{B}}$ TO $[f]_{\mathcal{a}}$ (FOR ALL $f \in V$) IS LINEAR, AND IS IN FACT AN ISOMORPHISM.

$$\begin{tikzcd}
 & V \arrow[dl, "\mathcal{B}\text{-COORDINATE TRANSFORMATION}"'] \arrow[dr, "\mathcal{A}\text{-COORDINATE TRANSFORMATION}"] & \\
\mathbb{R}^n \arrow[rr, "S"'] & & \mathbb{R}^n
\end{tikzcd}$$

LET $S = S_{\mathcal{B}\to\mathcal{A}}$ BE THE (INVERTIBLE, $n\times n$) MATRIX DEFINED BY

$$* \qquad S\,[f]_{\mathcal{B}} = [f]_{\mathcal{A}} \quad \text{FOR ALL } f \in V.$$

S IS CALLED THE CHANGE OF BASIS MATRIX FROM $\mathcal{B}$ TO $\mathcal{A}$. WE SEE FROM $*$ THAT

$$S_{\mathcal{A}\to\mathcal{B}} = S_{\mathcal{B}\to\mathcal{A}}^{-1}.$$

Also

$$S\,\vec{e}_j = S\,[f_j]_{\mathcal{B}} = [f_j]_{\mathcal{A}} \qquad (1 \le j \le n)$$

SO THAT

$$S_{\mathcal{B}\to\mathcal{A}} = \Big[\, [f_1]_{\mathcal{A}} \;\cdots\; [f_n]_{\mathcal{A}} \,\Big]$$

AND SIMILARLY

$$S_{\mathcal{A}\to\mathcal{B}} = \Big[\, [g_1]_{\mathcal{B}} \;\cdots\; [g_n]_{\mathcal{B}} \,\Big] .$$

LET $T: V \to V$ BE A LINEAR TRANSFORMATION, LET B BE THE MATRIX OF T WITH RESPECT TO $\mathcal{B}$, AND A BE ITS MATRIX WITH RESPECT TO $\mathcal{A}$. WHAT IS THE RELATIONSHIP BETWEEN B AND A?

$$\begin{tikzcd}
\mathbb{R}^n \arrow[rrr, "B"] \arrow[dd, "S"'] & & & \mathbb{R}^n \arrow[dd, "S = S_{\mathcal{B}\to\mathcal{A}}"] \\
& V \arrow[ul, "\mathcal{B}\text{-COORD. TRANS.}"'] \arrow[dl, "\mathcal{A}\text{-COORD. TRANS.}"] \arrow[r, "T"] & V \arrow[ur, "\mathcal{B}\text{-COORD. TRANS.}"] \arrow[dr, "\mathcal{A}\text{-COORD. TRANS.}"'] & \\
\mathbb{R}^n \arrow[rrr, "A"'] & & & \mathbb{R}^n
\end{tikzcd}$$

THE ABOVE DIAGRAM COMMUTES, BY THE DEFINITIONS OF THE VARIOS MAPS INVOLVED. THUS

$$SB = AS$$

i.e.

$$B = S^{-1} A S .$$

i.e. $B \sim A$.

THEOREM

TWO $n \times n$ MATRICES A, B ARE THE MATRICES OF A SINGLE LINEAR TRANSFORMATION $T: V \to V$ WITH RESPECT TO TWO BASES IF AND ONLY IF A IS SIMILAR TO B, WRITTEN $A \sim B$.

EX.

LET $\mathcal{B} = \{1, x, x^2\}$ AND $\mathcal{a} = \{2, x+1, x^2-1\}$. CHECK THAT BOTH $\mathcal{a}$ AND $\mathcal{B}$ ARE BASES OF $P_2 = \{$ Polynomials of degree $\leq 2 \}$ (EXERCISE.)

LET $D: P_2 \to P_2$ DENOTE THE DIFFERENTIATION OPERATION, i.e.

$$D(x^n) = n x^{n-1}$$

SO IN PARTICULAR

$$D(a + bx + cx^2) = b + 2cx .$$

AND

$$D(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$
$$D(x) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$
$$D(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$$

SO THE MATRIX OF D WITH RESPECT TO $\mathcal{B}$ IS

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Also

$$\begin{aligned} D(2) &= 0 = 0\cdot 2 + 0\cdot(x+1) + 0\cdot(x^2-1) \\ D(x+1) &= 1 = \tfrac{1}{2}\cdot 2 + 0\cdot(x+1) + 0\cdot(x^2-1) \\ D(x^2-1) &= 2x = (-1)\cdot 2 + 2(x+1) + 0\cdot(x^2-1) \end{aligned}$$

SO THE MATRIX OF D WITH RESPECT TO $\mathcal{A}$ IS

$$A = \begin{pmatrix} 0 & 1/2 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

TO FIND THE CHANGE OF BASIS MATRIX WE WRITE

$$\begin{aligned} 1 &= \tfrac{1}{2}\cdot 2 + 0\cdot(x+1) + 0\cdot(x^2-1) \\ x &= (-\tfrac{1}{2})\cdot 2 + 1\cdot(x+1) + 0\cdot(x^2-1) \\ x^2 &= (\tfrac{1}{2})\cdot 2 + 0\cdot(x+1) + 1\cdot(x^2-1) \end{aligned}$$

SO THAT

$$S = S_{\mathcal{B}\to\mathcal{A}} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

<u>EXERCISE</u>: VERIFY THAT

$$S^{-1} = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

AND THAT

$$B = S^{-1} A S \; .$$

HW (4.3) 2-10 even, 14, 18, 20, 22, 24,
40, 42, 44.