4.2 LINEAR TRANSFORMATIONS

LET V, W BE LINEAR SPACES. WE CALL A MAPPING $T : V \to W$ A <u>LINEAR TRANSFORMATION</u> IFF

a.) $T(f+g) = T(f) + T(g)$

AND

b.) $T(kf) = kT(f)$

FOR ALL $f, g \in V$, $k \in \mathbb{R}$.

THE IMAGE AND KERNEL OF T ARE DEFINED AS

$$\mathrm{im}(T) = \{T(f) \in W \mid f \in V\} \subseteq W$$
$$\mathrm{Ker}(T) = \{f \in V \mid T(f) = 0\} \subseteq V$$

<u>EXERCISE</u>

SHOW THAT $\mathrm{im}(T)$ IS A SUBSPACE OF W AND $\mathrm{Ker}(T)$ IS A SUBSPACE OF V.

WE DEFINE

$$\mathrm{rank}(T) = \dim(\mathrm{im}(T))$$

WHENEVER $\mathrm{im}(T)$ IS FINITE DIMENSIONAL. (OTHERWISE $\mathrm{rank}(T)$ IS UNDEFINED.)

LIKEWISE WE DEFINE

$$\text{nullity}(T) = \dim(\ker(T))$$

IFF KER(T) IS FINITE DIMENSIONAL.

EX. LET $D: C^{\infty} \to C^{\infty}$ BE THE DIFFERENTIATION OPERATOR. THAT D IS LINEAR IS A BASIC FACT OF ELEMENTARY CALCULUS.

$$\ker(D) = \{\text{CONSTANT FUNCTIONS}\}$$

AND

$$\text{im}(D) = C^{\infty}.$$

THIS LAST EQUATION FOLLOWS FROM THE FUNDAMENTAL THEOREM OF CALCULUS, WHICH CAN BE STATED AS

$$D \int_0^x f(t)\,dt = f(x)$$

SHOWING THAT D IS SURJECTIVE.

EX. FIX ANY $A \in M_n$ AND DEFINE $T: M_n \to M_n$ BY

$$T(B) = AB$$

ONE CHECKS EASILY THAT T is linear. Also

$B \in \text{Ker}(T)$

$\iff T(B) = 0$

$\iff AB = 0$

$\iff [A\vec{v}_1 \cdots A\vec{v}_n] = 0$ WHERE $B = [\vec{v}_1 \cdots \vec{v}_n]$

$\iff A\vec{v}_1 = \cdots = A\vec{v}_n = \vec{0}$

$\iff \vec{v}_1, \ldots, \vec{v}_n \in \text{Ker}(A)$

$\iff \text{span}(\vec{v}_1, \ldots, \vec{v}_n) \subseteq \text{Ker}(A)$

$\iff \text{im} B \subseteq \text{Ker}(A)$.

$$\therefore \ \text{Ker}(T) = \{ B \in M_n \mid \text{im} B \subseteq \text{Ker} A \}$$

Similarly

$C \in \text{im}(T)$

$\iff C = T(B)$ FOR SOME $B = [\vec{v}_1 \cdots \vec{v}_n] \in M_n$

$\iff C = AB = [A\vec{v}_1 \cdots A\vec{v}_n]$

$\iff \{\text{Columns of } C\} \subseteq \text{im} A$

$$\therefore \ \text{Im}(T) = \{ C \in M_n \mid \{\text{Columns of } C\} \subseteq \text{im} A \}$$

DEFN

A LINEAR TRANSFORMATION $T: V \to W$ WHICH IS INVERTIBLE IS CALLED AN *ISOMORPHISM*. IN THIS CASE WE SAY V AND W ARE *ISOMORPHIC* AND WE WRITE

$$V \approx W$$

ISOMORPHIC LINEAR SPACES ARE THE SAME IN EVERY RESPECT THAT MATTERS IN LINEAR ALGEBRA. OBSERVE THAT ISOMORPHISM IS AN EQUIVALENCE RELATION IN THE SENSE DISCUSSED ON PAGES 128-129 OF THE NOTES.

- REFLEXIVE : $V \approx V$
- SYMMETRIC : $V \approx W \Rightarrow W \approx V$
- TRANSITIVE : $V \approx W$ AND $W \approx U \Rightarrow V \approx U$.

<u>EXERCISE</u>

PROVE THAT THE ISOMORPHISM RELATION $\approx$ SATISFIES EACH OF THE ABOVE PROPERTIES.

<u>EX.</u>

LET $\mathcal{B} = \{f_1, .., f_n\}$ BE A BASIS OF A LINEAR SPACE V. THE $\mathcal{B}$-COORDINATE TRANSFORMATION FROM $V \to \mathbb{R}^n$ TAKING

$$f \longrightarrow [f]_{\mathcal{B}}$$

i.e.

$$c_1 f_1 + \cdots + c_n f_n \longrightarrow \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

IS AN ISOMORPHISM, SINCE IT IS BOTH LINEAR, AND INVERTIBLE. THIS PROVES

THEOREM

IF V IS A FINITE DIMENSIONAL LINEAR SPACE, SAY $\dim V = n$, THEN

$$V \cong \mathbb{R}^n$$

i.e. $\mathbb{R}^n$ IS IN A SENSE, THE only linear SPACE OF DIMENSION n.

THEOREM

LET $T : V \to W$ BE LINEAR, THEN

a.) T IS AN ISOMORPHISM IFF BOTH $\ker(T) = \{0\}$ AND $\operatorname{im}(T) = W$

FOR b AND c ASSUME V, W ARE FINITE DIMENSIONAL

b.) IF T IS AN ISOMORPHISM, THEN $\dim V = \dim W$

c.) IF $\ker(T) = \{0\}$ AND $\dim V = \dim W$, THEN T IS AN ISOMORPHISM.

PROOF: FIRST OBSERVE THAT $\ker(T) = \{0\}$

IFF FOR ANY $f \in V$: $T(f) = 0 \Rightarrow f = 0$

IFF FOR ANY $f, g \in V$: $T(f-g) = 0 \Rightarrow f - g = 0$

IFF FOR ANY $f, g \in V$: $T(f) = T(g) \Rightarrow f = g$

IFF T IS INJECTIVE.

THUS $\ker(T)=\{0\}$ AND $\operatorname{im}(T)=W$ IFF T IS INJECTIVE AND T IS SURJECTIVE IFF T IS INVERTIBLE IFF T IS AN ISOMORPHISM.

TO PROVE (b) WE SHOW THAT IF $\{f_1,..,f_n\}$ IS A BASIS OF V, THEN $\{T(f_1),...,T(f_n)\}$ IS A BASIS OF W.

LET $\{f_1,..,f_n\}$ BE A BASIS OF V, AND SUPPOSE $c_1T(f_1)+\cdots+c_nT(f_n)=0$. THEN $T(c_1f_1+\cdots+c_nf_n)=0$, WHENCE $c_1f_1+\cdots+c_nf_n=0$, SINCE T IS INJECTIVE. THEN $c_1=\cdots=c_n=0$ SINCE $\{f_1,..,f_n\}$ IS LINEARLY INDEPENDENT. THEREFORE $\{T(f_1),..,T(f_n)\}$ IS ALSO LINEARLY INDEPENDENT. NOW PICK ANY $g\in W$, AND LET $f=T^{-1}(g)$. THERE EXIST $c_1,..,c_n \in\mathbb{R}$ SUCH THAT $f=c_1f_1,..,c_nf_n$, SINCE $\{f_1,..,f_n\}$ SPANS V. THUS $g=T(f) = c_1T(f_1)+\cdots+c_nT(f_n)$, SHOWING THAT $g\in \operatorname{SPAN}(T(f_1),..,T(f_n))$, WHENCE $\{T(f_1),..,T(f_n)\}$ SPANS W.

PART (c) FOLLOWS FROM THE FACT THAT $T:\mathbb{R}^n\to\mathbb{R}^n$ IS INJECTIVE IFF SURJECTIVE (SEE P. 84 OF NOTES.)

HW (4.2) 2-16 even, 34, 56, 52