2.2 GEOMETRY OF LINEAR TRANSFORMATIONS

LET $T: \mathbb{R}^2 \to \mathbb{R}^2$ BE GIVEN BY THE MATRIX

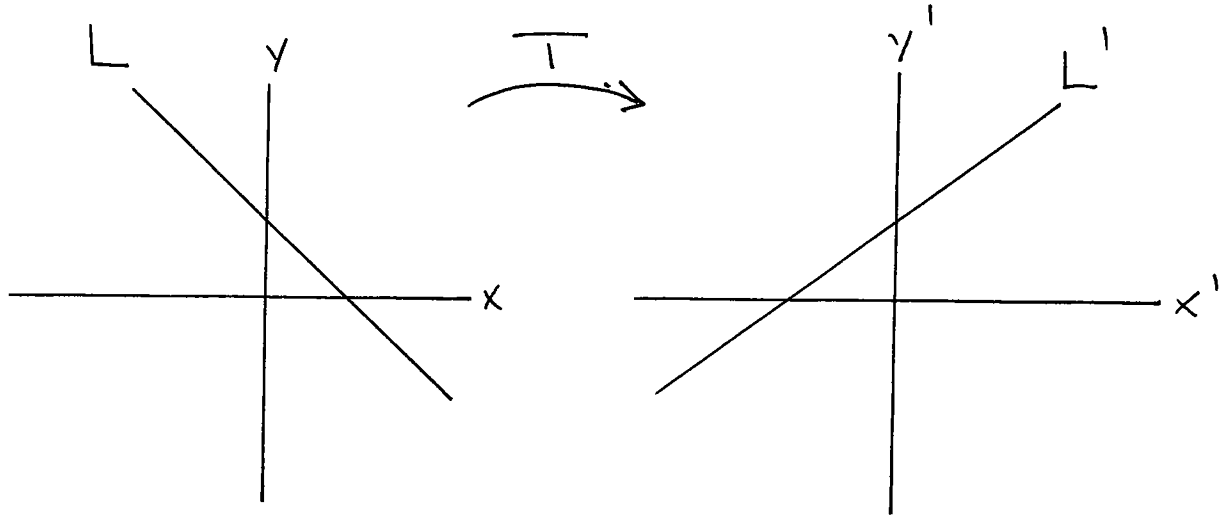
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

WE WILL USE COORDINATES $\begin{pmatrix} x \\ y \end{pmatrix}$ FOR THE DOMAIN $\mathbb{R}^2$ AND $\begin{pmatrix} x' \\ y' \end{pmatrix}$ FOR THE CODOMAIN $\mathbb{R}^2$. THUS T IS

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

i.e.

$$T: \begin{cases} x' = ax + by \\ y' = cx + dy \end{cases}$$



$\mathbb{R}^2$: DOMAIN $\qquad\qquad$ $\mathbb{R}^2$: CODOMAIN

ONE MAY VERIFY THAT T CARRIES POINTS ON THE LINE

$$L: \ \alpha x + \beta y = \gamma$$

IN THE $\binom{x}{y}$-PLANE INTO POINTS ON THE LINE

$$L': \ (d\alpha - c\beta)x' + (a\beta - b\alpha)y' = (ad-bc)\gamma$$

IN THE $\binom{x'}{y'}$ PLANE.

<u>CHECK:</u>

$$\begin{aligned}
&(d\alpha - c\beta)(ax+by) + (a\beta - b\alpha)(cx+dy) \\
&= ad\alpha x - \cancel{ac\beta x} + \cancel{bd\alpha y} - bc\beta y + \cancel{ac\beta x} - bc\alpha x + ad\beta y - \cancel{bd\alpha y} \\
&= (ad-bc)\alpha x + (ad-bc)\beta y \\
&= (ad-bc)(\alpha x + \beta y) \\
&= (ad-bc)\gamma \qquad \text{SINCE } \tbinom{x}{y} \text{ LIES ON } L.
\end{aligned}$$

THUS IN GENERAL T TAKES LINES TO LINES.

THUS TO FIND THE IMAGE OF A LINE, ONE NEEDS ONLY COMPUTE THE IMAGES OF TWO DISTINCT POINTS ON THAT LINE.

Ex. $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2y \end{pmatrix}$

$L: \; x + 2y = 7$ PARAMETRIC EQNS: $\begin{pmatrix} 7-2t \\ t \end{pmatrix}$

$L': \; 2x' + y' = 14$ " $\begin{pmatrix} 7-t \\ 2t \end{pmatrix}$

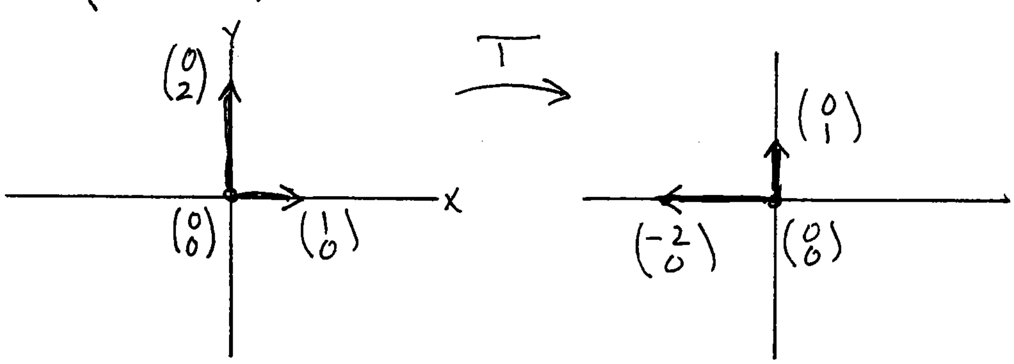
Ex. IF IT SO HAPPENS THAT $ad-bc=0$, THEN T will TAKE SOME lines TO A SINGLE POINT

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

$L: \; x = 6$ $L': \; \left\{ \begin{pmatrix} 6 \\ 0 \end{pmatrix} \right\}$

THIS SITUATION DOES NOT ARISE IF $ad-bc \neq 0$ (PROOF LATER.)

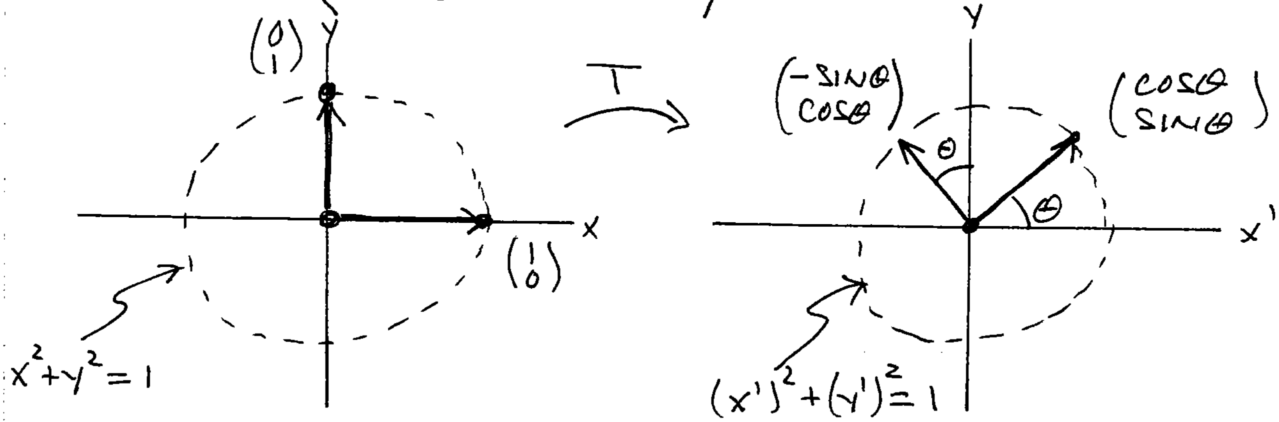
Ex. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $T\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$



THUS T is A COUNTERCLOCKWISE ROTATION OF THE PLANE THROUGH $\pi/2$ RADIANS $(90°)$.

EX

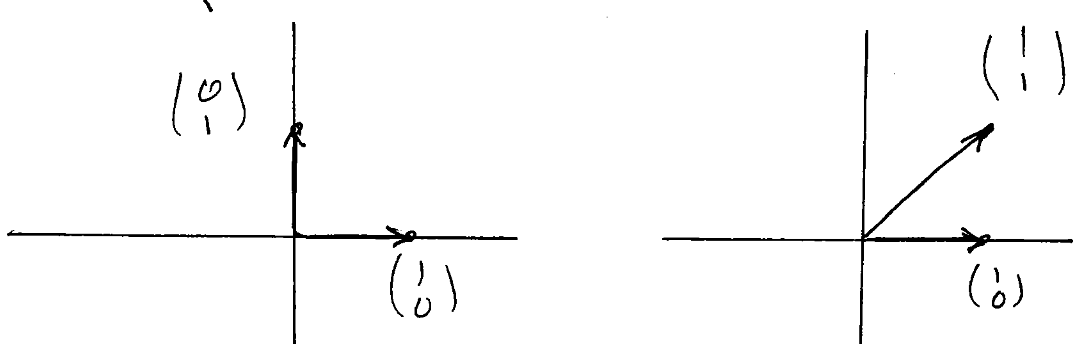
$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$



CHECK: T TAKES POINTS ON THE UNIT CIRCLE TO POINTS ON THE UNIT CIRCLE.

EX

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

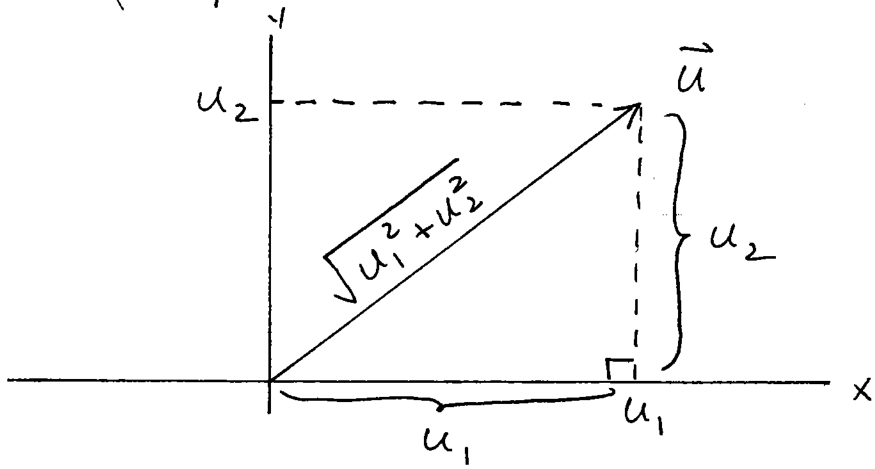


THIS OPERATION IS KNOWN AS A SHEAR.
IN GENERAL

$\begin{pmatrix} 1 & K \\ 0 & 1 \end{pmatrix}$ REPRESENTS A SHEAR PARALLEL TO $\begin{pmatrix}1\\0\end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ K & 1 \end{pmatrix}$ REPRESENTS A SHEAR PARALLEL TO $\begin{pmatrix}0\\1\end{pmatrix}$.

LET $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^2$



BY THE PYTHAGOREAN THEOREM, THE <u>LENGTH</u> OF $\vec{u}$, DENOTED $|\vec{u}|$ IS

$$|\vec{u}| = \sqrt{u_1^2+u_2^2}$$

SO

$$|\vec{u}|^2 = u_1^2+u_2^2 = \vec{u} \cdot \vec{u}$$

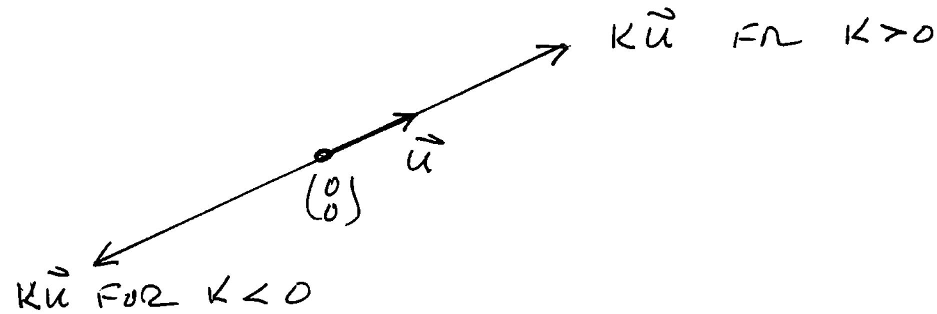
IF $k \in \mathbb{R}$ THEN $\vec{u} = \begin{pmatrix} ku_1 \\ ku_2 \end{pmatrix}$ AND

$$|k\vec{u}| = \sqrt{(ku_1)^2+(ku_2)^2} = \sqrt{k^2(u_1^2+u_2^2)} = |k|\sqrt{u_1^2+u_2^2}$$

$$= |k| \cdot |\vec{u}|$$

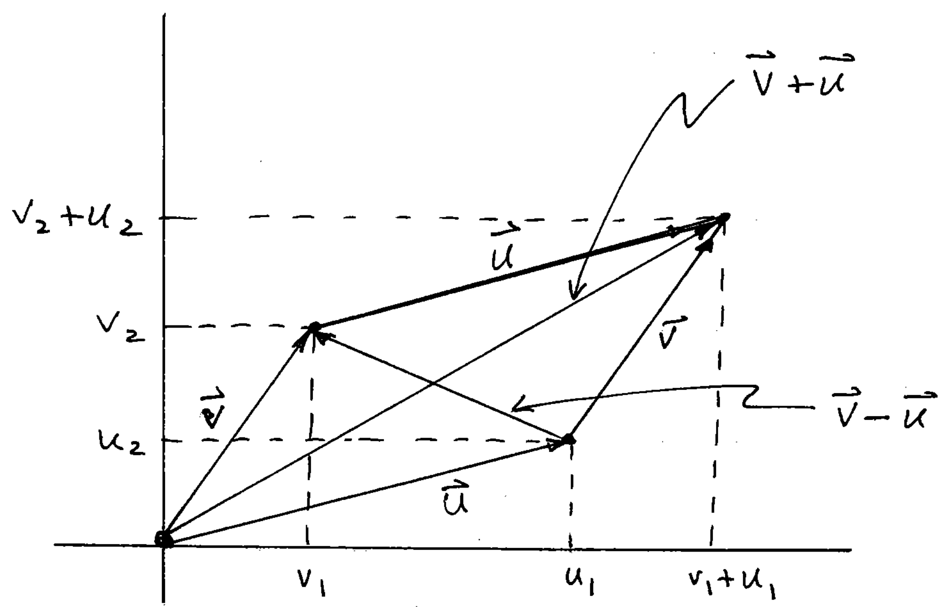
THUS IF $k > 0$, MULTIPLICATION BY THE SCALAR k HAS THE EFFECT OF EXPANDING THE LENGTH OF $\vec{u}$ BY A FACTOR OF k

IF $k < 0$, THE OPERATION RESCALES AND REVERSES DIRECTION



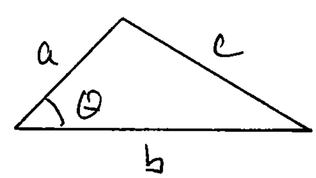
IF $k=0$ THEN $k\vec{u}=\begin{pmatrix}0\\0\end{pmatrix}$ which has length 0.

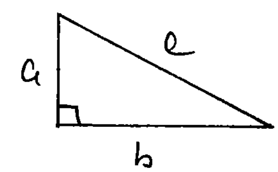
NOW LET $\vec{v}=\begin{pmatrix}v_1\\v_2\end{pmatrix}\in\mathbb{R}^2$. THEN $\vec{v}+\vec{u}=\begin{pmatrix}v_1+u_1\\v_2+u_2\end{pmatrix}$ is THE VECTOR WITH TAIL AT $\binom{0}{0}$ AND ARROW AT THE OPPOSITE VERTEX OF THE PARALLELOGRAM SPANNED BY $\vec{u}$ AND $\vec{v}$:



THE DIFFERENCE $\vec{v}-\vec{u}=\begin{pmatrix}v_1-u_1\\v_2-u_2\end{pmatrix}$ is THE VECTOR WITH TAIL AT THE ARROW OF $\vec{u}$ AND ARROW AT THE ARROW OF $\vec{v}$.

RECALL THE <u>LAW OF COSINES</u> FROM TRIGONOMETRY



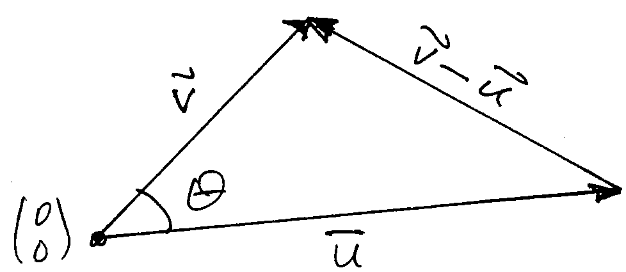


SPECIAL CASE: $\theta = \frac{\pi}{2}$

$$a^2+b^2 = c^2+2ab\cos\theta \qquad\qquad a^2+b^2=c^2$$

TH REDUCES TO THE PYTHAGOREAN THEOREM IN THE SPECIAL CASE $\theta = \frac{\pi}{2}$, FOR THEN $\cos\theta = 0$.

NOW LET $\vec{u}, \vec{v} \in \mathbb{R}^2$ AS BEFORE, AND LET θ BE THE ANGLE BETWEEN THEM



THEN THE VECTORS $\vec{u}, \vec{v}, \vec{v}-\vec{u}$ FORM A TRIANGLE TO WHICH WE CAN APPLY THE LAW OF COSINES

$$|\vec{v}|^2+|\vec{u}|^2 = |\vec{v}-\vec{u}|^2 + 2|\vec{u}||\vec{v}|\cos\theta$$

i.e.

$$(v_1^2+v_2^2)+(u_1^2+u_2^2) = (v_1-u_1)^2+(v_2-u_2)^2+2|\vec{u}||\vec{v}|\cos\theta$$

$$\therefore\ \cancel{v_1^2}+\cancel{v_2^2}+\cancel{u_1^2}+\cancel{u_2^2} = \cancel{v_1^2}-2u_1v_1+\cancel{u_1^2}+\cancel{v_2^2}-2u_2v_2+\cancel{u_2^2}+|\vec{u}||\vec{v}|\cos\theta$$

$$\therefore \quad 2u_1v_1 + 2u_2v_2 = 2|\vec{u}||\vec{v}|\cos\theta$$

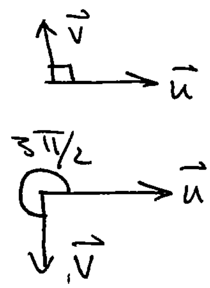
$$\therefore \quad u_1v_1 + u_2v_2 = |\vec{u}||\vec{v}|\cos\theta$$

i.e. $$\boxed{\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta}$$

NOTE: THIS IMPORTANT INTERPRETATION OF THE DOT PRODUCT READILY EXTENDS TO HIGHER DIMENSIONS.

WE WRITE $\vec{u} \perp \vec{v}$ TO MEAN THAT $\vec{u}$ IS PERPENDICULAR TO $\vec{v}$. OBSERVE

$$\vec{u} \perp \vec{v} \iff \begin{cases} \theta = \frac{\pi}{2} \\ \text{or} \\ \theta = \frac{3\pi}{2} \end{cases}$$



$$\iff \cos\theta = 0$$

$$\iff \vec{u}\cdot\vec{v} = 0$$

i.e. $$\boxed{\vec{u} \perp \vec{v} \quad \text{IFF} \quad \vec{u}\cdot\vec{v} = 0}$$

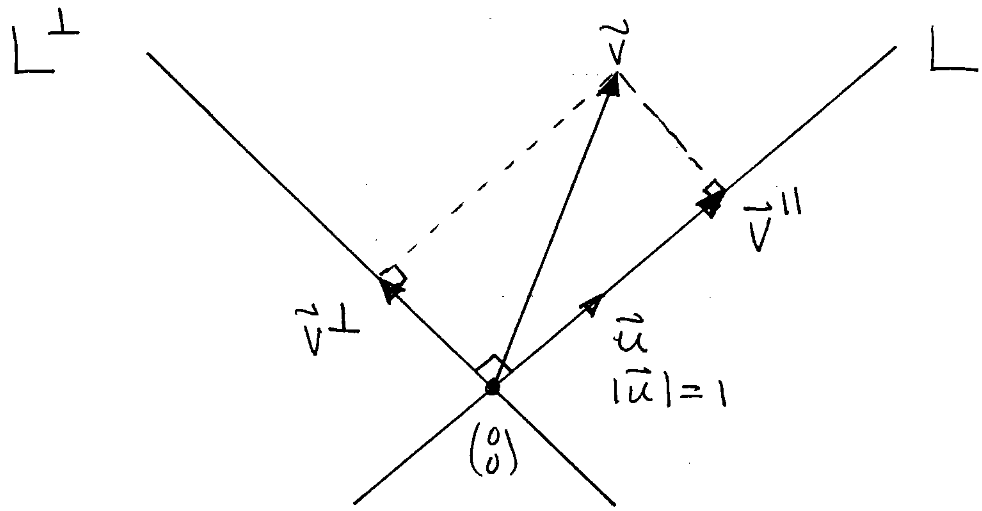
EXERCISE LET $A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ BE THE ROTATION MATRIX DISCUSSED IN AN EARLIER EXAMPLE. SHOW THAT FOR ANY $\vec{u}, \vec{v} \in \mathbb{R}^2$ THAT

$$(A_\theta\vec{u})\cdot(A_\theta\vec{v}) = \vec{u}\cdot\vec{v}$$

i.e. SHOW THAT THE DOT PRODUCT IS ROTATION INVARIANT. (HINT: USE THE IDENTITY $\cos^2\theta + \sin^2\theta = 1$.)

PROJECTIONS

LET L BE A LINE THROUGH THE ORIGIN $\begin{pmatrix}0\\0\end{pmatrix}$ IN $\mathbb{R}^2$, AND LET $L^\perp$ BE THE LINE THROUGH THE ORIGIN WHICH IS PERPENDICULAR TO L.



ANY VECTOR $\vec{v} \in \mathbb{R}^2$ CAN BE DECOMPOSED (UNIQUELY) INTO A SUM

$$\vec{v} = \vec{v}^{\parallel} + \vec{v}^{\perp}$$

WHERE $\vec{v}^{\parallel}$ IS PARALLEL TO L AND $\vec{v}^{\perp}$ IS PARALLEL TO $L^\perp$.

THE <u>ORTHOGONAL PROJECTION OF $\vec{v}$ ONTO L</u> IS THE VECTOR

$$\mathrm{Proj}_L(\vec{v}) = \vec{v}^{\parallel}$$

IT IS NOT IMMEDIATELY CLEAR THAT THE MAPPING

$$\text{Proj}_L : \mathbb{R}^2 \to \mathbb{R}^2$$

IS A LINEAR TRANSFORMATION, BUT INDEED IT IS.

TO SEE THIS LET $\vec{u}$ BE A UNIT VECTOR IN THE DIRECTION L (i.e. $|\vec{u}|=1$) AND OBSERVE THAT

$$\text{Proj}_L(\vec{v}) = \vec{v}^{\parallel} = k\vec{u}$$

FOR SOME CONSTANT $k \in \mathbb{R}$. NOW NOTE THAT

$$\vec{v}^{\perp} = \vec{v} - \vec{v}^{\parallel} = \vec{v} - k\vec{u}$$

IS PERPENDICULAR TO $\vec{u}$, SO THAT

$$0 = (\vec{v} - k\vec{u}) \cdot \vec{u} = (\vec{v} \cdot \vec{u}) - k\underbrace{(\vec{u} \cdot \vec{u})}_{1} = (\vec{v} \cdot \vec{u}) - k$$

$$\therefore \quad k = (\vec{v} \cdot \vec{u})$$

$$\therefore \quad \boxed{\text{Proj}_L(\vec{v}) = (\vec{v} \cdot \vec{u})\,\vec{u}}$$

THIS FORMULA CAN BE USED TO PROVE THAT Proj_L IS A LINEAR TRANSFORMATION.

EXERCISE

LET $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$, $s, t \in \mathbb{R}$ AND PROVE THE FOLLOWING IDENTITIES

(1) $\vec{x} \circ \vec{y} = \vec{y} \circ \vec{x}$
(2) $(\vec{x} + \vec{y}) \circ \vec{z} = (\vec{x} \circ \vec{z}) + (\vec{y} \circ \vec{z})$
(3) $s(t\vec{x}) = (st)\vec{x}$
(4) $(s+t)\vec{x} = (s\vec{x}) + (t\vec{x})$
(5) $s(\vec{x} \circ \vec{y}) = (s\vec{x}) \circ \vec{y} = \vec{x} \circ (s\vec{y})$.

EXERCISE

USE THE PRECEDING IDENTITIES TO SHOW FOR ALL $\vec{v}, \vec{w} \in \mathbb{R}^2$, $k \in \mathbb{R}$ THAT

- $\mathrm{Proj}_L(\vec{v} + \vec{w}) = \mathrm{Proj}_L(\vec{v}) + \mathrm{Proj}_L(\vec{w})$

and

- $\mathrm{Proj}_L(k\vec{v}) = k\,\mathrm{Proj}_L(\vec{v})$

THUS $\mathrm{Proj}_L : \mathbb{R}^2 \to \mathbb{R}^2$ IS A LINEAR TRANSFORMATION. ITS MATRIX IS

$$A = \left[\mathrm{Proj}_L\begin{pmatrix}1\\0\end{pmatrix} \quad \mathrm{Proj}_L\begin{pmatrix}0\\1\end{pmatrix} \right]$$

$$= \left[\left(\begin{pmatrix}1\\0\end{pmatrix} \circ \begin{pmatrix}u_1\\u_2\end{pmatrix}\right)\begin{pmatrix}u_1\\u_2\end{pmatrix} \quad \left(\begin{pmatrix}0\\1\end{pmatrix} \circ \begin{pmatrix}u_1\\u_2\end{pmatrix}\right)\begin{pmatrix}u_1\\u_2\end{pmatrix} \right]$$

$$= \left[u_1\begin{pmatrix}u_1\\u_2\end{pmatrix} \quad u_2\begin{pmatrix}u_1\\u_2\end{pmatrix} \right] = \begin{pmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{pmatrix}.$$

THUS ANY LINE L (OR ANY UNIT VECTOR $\vec{u}$) DEFINES A LINEAR TRANSFORMATION $\mathbb{R}^2 \to \mathbb{R}^2$.

<u>EXERCISE</u>

WRITE A FORMULA FOR THE PROJECTION OF A VECTOR $\vec{v}$ ONTO $L^{\perp}$, SHOW THAT THIS MAPPING IS LINEAR, AND DETERMINE ITS MATRIX.

IN A SIMILAR MANNER WE CAN DEFINE PROJECTIONS ONTO LINES AND PLANES IN $\mathbb{R}^3$. (SEE P. 60-61.)

HW (2.2) 2, 4, 6, 8, 10, 12, 14ab, 20, 22, 26abcd
28abcde, 30.