1.2 MATRICIES

RECALL OUR FIRST EXAMPLE

$$\begin{cases} 2x + 8y + 4z = 2 \\ 2x + 5y + z = 5 \\ 4x + 10y - z = 1 \end{cases}$$

THE ONLY RELEVANT INFORMATION IN THIS SYSTEM IS THE COEFFICIENTS AND THE NUMBERS ON THE RIGHT HAND SIDE.

AUGMENTED MATRIX

$$\begin{pmatrix} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{pmatrix} \qquad 3 \times 4$$

COEFFICIENT MATRIX

$$\begin{pmatrix} 2 & 8 & 4 \\ 2 & 5 & 1 \\ 4 & 10 & -1 \end{pmatrix} \qquad 3 \times 3$$

AN $n \times m$ MATRIX HAS n ROWS AND m COLUMNS. ROW VECTOR: $1 \times n$

COLUMN VECTOR: $n \times 1$

LETS SOLVE THAT FIRST SYSTEM AGAIN USING THE AUGMENTED MATRIX.

Ex.

$$\begin{pmatrix} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{pmatrix} \begin{array}{l} \div 2 \\ \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 4 & 2 & 1 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{pmatrix} \begin{array}{l} \\ -2\cdot \mathrm{I} \\ -4\cdot \mathrm{I} \end{array}$$

$$\begin{pmatrix} 1 & 4 & 2 & 1 \\ 0 & -3 & -3 & 3 \\ 0 & -6 & -9 & -3 \end{pmatrix} \begin{array}{l} \\ \div(-3) \\ \div(-3) \end{array}$$

$$\begin{pmatrix} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 3 & 1 \end{pmatrix} \begin{array}{l} -4\cdot \mathrm{II} \\ \\ -2\cdot \mathrm{II} \end{array}$$

$$\begin{pmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix} \begin{array}{l} +2\cdot \mathrm{III} \\ -1\cdot \mathrm{III} \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{pmatrix} \longrightarrow \begin{array}{l} x = 11 \\ y = -4 \\ z = 3 \end{array}$$

Ex.

$$\begin{cases} x_1 + 2x_3 + 4x_4 = -8 \\ x_2 - 3x_3 - x_4 = 6 \\ 3x_1 + 4x_2 - 6x_3 + 8x_4 = 0 \\ -x_2 + 3x_3 + 4x_4 = -12 \end{cases}$$

$$\begin{array}{cl} \begin{pmatrix} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 3 & 4 & -6 & 8 & 0 \\ 0 & -1 & 3 & 4 & -12 \end{pmatrix} & \begin{array}{l} \\ \\ -3\cdot \mathrm{I} \\ \\ \end{array} \end{array}$$

$$\begin{array}{cl} \begin{pmatrix} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 4 & -12 & -4 & 24 \\ 0 & -1 & 3 & 4 & -12 \end{pmatrix} & \begin{array}{l} \\ \\ -4\cdot \mathrm{II} \\ +1\cdot \mathrm{II} \end{array} \end{array}$$

$$\begin{pmatrix} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & -6 \end{pmatrix}$$

$$\begin{array}{cl} \begin{pmatrix} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 3 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{l} \\ \\ \div 3 \\ \\ \end{array} \end{array}$$

$$\begin{pmatrix} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} -4\cdot \mathrm{III} \\ +1\cdot \mathrm{III} \\ \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 \quad + 2x_3 = 0 \\ x_2 - 3x_3 = 4 \\ x_4 = -2 \\ 0 = 0 \end{cases}$$

LET $x_3 = t \in \mathbb{R}$ BE ARBITRARY. THEN $x_1 = -2t$, $x_2 = 3t + 4$. THUS THE SOLUTIONS ARE

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2t \\ 3t+4 \\ t \\ -2 \end{pmatrix}$$

GEOMETRICALLY, THIS IS A LINE IN 4-DIMENSIONAL SPACE.

A MATRIX IS IN <u>REDUCED ROW-ECHELON FORM</u> IF IT SATISFIES THE FOLLOWING

a.) IF A ROW CONTAINS NON-ZERO ENTRIES, THEN THE FIRST (i.e. LEFTMOST) NON-ZERO ENTRY IN THAT ROW IS 1. WE CALL THIS THE <u>LEADING 1</u> IN THAT ROW.

b.) IF A COLUMN CONTAINS A LEADING 1, THEN ALL OTHER ENTRIES IN THAT COLUMN ARE 0.

c.) IF A ROW CONTAINS A LEADING 1, THEN EACH ROW ABOVE IT CONTAINS A LEADING 1 FURTHER TO THE LEFT

THE LAST MATRIX IN THE PRECEDING EXAMPLE WAS IN RREF.

OUR GOAL THEN IS TO BRING AN AUGMENTED MATRIX INTO RREF BY PERFORMING <u>ELEMENTARY ROW OPERATIONS</u>.

1.) SWAP TWO ROWS
2.) MULTIPLY OR DIVIDE A ROW BY SOME CONSTANT.
3.) ALTER A ROW BY ADDING A CONSTANT MULTIPLE OF SOME OTHER ROW TO IT.

OBSERVE THAT EACH OF THE ELEMENTARY ROW OPERATIONS IS INVERTIBLE IN THE SENSE THAT ITS EFFECT ON THE SYSTEM CAN BE REVERSED BY ANOTHER ERO OF THE SAME TYPE.

CONSIDER THE 2×3 SYSTEM:

A
$$\begin{cases} ax + by + cz = d \\ ex + fy + gz = h \end{cases} \qquad [+\alpha \mathrm{I}] \longleftarrow \text{ERO.}$$

INVERSE ERO.

B
$$\begin{cases} ax + by + cz = d \\ (\alpha a + e)x + (\alpha b + f)y + (\alpha c + g)z = \alpha d + h \end{cases} \qquad [-\alpha \mathrm{I}]$$

$$\begin{cases} ax + by + cz = d \\ (-\cancel{\alpha a} + \cancel{\alpha a} + e)x + (-\cancel{b\alpha} + \cancel{b\alpha} + f)y + (-\cancel{\alpha c} + \cancel{\alpha c} + g)z = -\cancel{\alpha d} + \cancel{\alpha d} + h \end{cases}$$

A
$$\begin{cases} ax + by + cz = d \\ ex + fy + gz = h \end{cases}$$

ONE CAN SEE NOW WHY AN ERO OF TYPE (3) PRESERVES THE SOLUTION SET OF A LINEAR SYSTEM. ANY SOLUTION (x, y, z) TO SYSTEM A MUST ALSO SATISFY SYSTEM B SINCE "EQUALS ADDED TO EQUALS GIVE EQUALS". LIKEWISE ANY SOLUTION TO B ALSO SOLVES A.

GEOMETRICALLY, THE PLANE REPRESENTED BY EQN. A.II IS TRADED IN FOR ANOTHER PLANE, NAMELY THAT REPRESENTED BY B.II,

HAVING THE SAME LINE OF INTERSECTION WITH PLANE A.I. IN OTHER WORDS, PLANE A.II IS ROTATED ABOUT ITS LINE OF INTERSECTION WITH A.I UNTIL IT COINCIDES WITH PLANE B.II.

EROs (1) AND (2) LEAVE THE PLANES THEMSELVES UNCHANGED, AND SO THESE OPERATIONS ALSO PRESERVE THE SOLUTION SET.

WE CAN NOW STATE THE GAUSS-JORDAN ELIMINATION ALGORITHM FORMALLY

<u>GAUSS-JORDAN ELIMINATION (A)</u>

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        1.)  n ← #ROWS[A] , m ← #COLS[A]
        2.)  i ← 1 , j ← 1
        3.)  while i ≤ n and j ≤ m
        4.)      look for a non-zero entry in col j at or
                 below row i
        5.)      if such an entry is found, say at row k
ERO(1)  6.)          if k ≠ i swap row i ↔ row k
ERO(2)  7.)          if a_ij ≠ 1 Divide row i by a_ij
ERO(3)  8.)          Eliminate all other non-zero entries
                     from col j by SUBTRACTING from
                     each row (other than i) an
                     appropriate multiple of row i
        9.)          i ← i+1
        10.)     j ← j+1
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A MOMENT'S THOUGHT SHOULD CONVINCE YOU THAT THE MATRIX WHICH RESULTS FROM PERFORMING THIS ALGORITHM WILL BE IN REDUCED ROW ECHELON FORM. WE DENOTE THIS RESULTANT MATRIX BY:

$$\mathrm{RREF}(A)$$

<u>DEFN</u>

THE <u>RANK</u> OF A MATRIX A IS THE NUMBER OF LEADING 1s IN $\mathrm{RREF}(A)$, DENOTED $\mathrm{rank}(A)$.

OBSERVE THAT EACH TIME THE CONDITION IN STEP 5 IS TRUE, A NEW LEADING 1 IS CREATED. THUS THE ALGORITHM CAN BE EASILY MODIFIED TO RETURN THE RANK BY JUST ADDING THE INSTRUCTION

11.) return $(i-1)$

OUTSIDE THE while loop.

Ex.

$$\begin{cases} x_1 - 7x_2 + 2x_4 + 3x_5 = 5 \\ 2x_1 - 14x_2 + x_3 + x_4 + x_5 = 9 \\ x_4 + x_5 = 1 \end{cases}$$

$$\begin{pmatrix} 1 & -7 & 0 & 2 & 3 & 5 \\ 2 & -14 & 1 & 1 & 1 & 9 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{array}{l} \\ -2\cdot \mathrm{I} \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & -7 & 0 & 2 & 3 & 5 \\ 0 & 0 & 1 & -3 & -5 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{array}{l} -2\cdot \mathrm{III} \\ +3\cdot \mathrm{III} \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & -7 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & -2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{cases} x_1 - 7x_2 + x_5 = 3 \\ x_3 - 2x_5 = 2 \\ x_4 + x_5 = 1 \end{cases}$$

LET $x_2 = t$, $x_5 = s$ BE ARBITRARY

THEN

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 7t - s + 3 \\ t \\ 2s + 2 \\ -s + 1 \\ s \end{pmatrix}$$

EXERCISE: CHECK SOLNS.

THE SOLUTION SET IN THIS EXAMPLE HAS 2 FREE VARIABLES ($x_2 = t$, $x_5 = s$). NOTE THAT THESE ARE PRECISELY THE VARIABLES WHOSE COLUMNS DO NOT CONTAIN A LEADING 1.

HW (1.2)

2, 4, 6, 8, 10, 12, 16, 18abcd, 20, 22, 34, 46.