LINEAR ALGEBRA
WINTER 2007

1.1 SYSTEMS OF LINEAR EQUATIONS

EX

$$\begin{cases} 2x + 8y + 4z = 2 & \text{I} \\ 2x + 5y + z = 5 & \text{II} \\ 4x + 10y - z = 1 & \text{III} \end{cases}$$

GOAL: FIND ALL TRIPLES (x, y, z) WHICH SIMULTANEOUSLY SATISFY I, II, AND III.

TECHNIQUE: PERFORM A SEQUENCE OF OPERATIONS THAT CONVERT THE SYSTEM INTO A SIMPLER ONE, WHICH IS EQUIVALENT TO THE ORIGINAL.

- DIVIDE I BY 2:

$$\begin{cases} x + 4y + 2z = 1 \\ 2x + 5y + z = 5 \\ 4x + 10y - z = 1 \end{cases}$$

- ADD (-2)I TO II, ADD (-4)I TO III

$$\begin{cases} x + 4y + 2z = 1 \\ \quad -3y - 3z = 3 \\ \quad -6y - 9z = -3 \end{cases}$$

- DIVIDE II BY -3, DIVIDE III BY -3

$$\begin{cases} x + 4y + 2z = 1 \\ y + z = -1 \\ 2y + 3z = 1 \end{cases}$$

- ADD (-4)II TO I, ADD (-2)II TO III

$$\begin{cases} x - 2z = 5 \\ y + z = -1 \\ z = 3 \end{cases}$$

- ADD (-1)III TO II, ADD $2 \cdot$III TO I

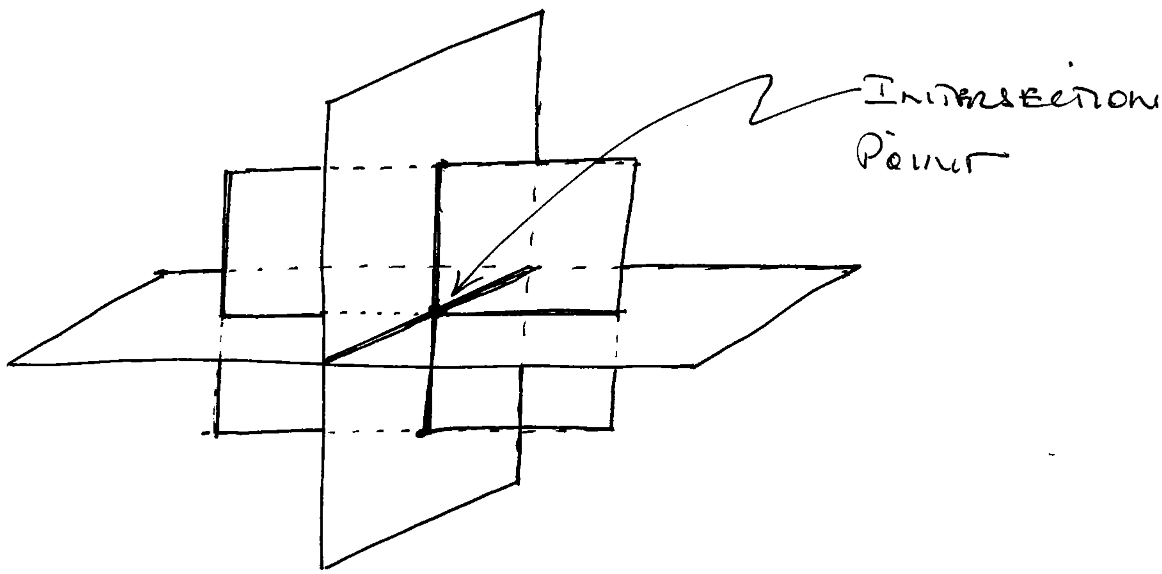
$$\begin{cases} x = 11 \\ y = -4 \\ z = 3 \end{cases}$$

THIS SYSTEM IS EQUIVALENT TO THE ORIGINAL (IN THE SENSE THAT IT HAS EXACTLY THE SAME SOLUTIONS.)

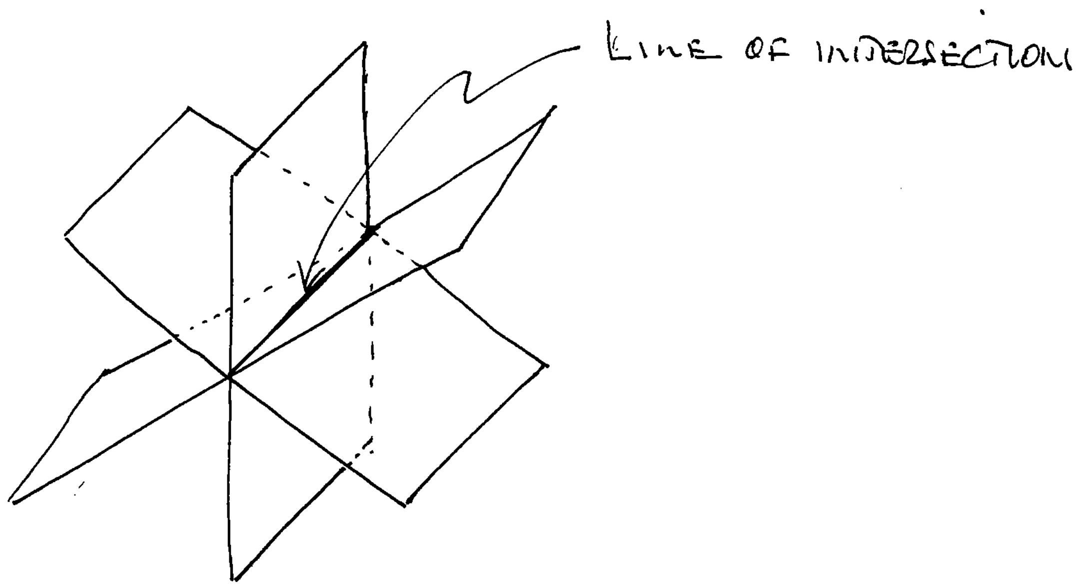
BUT THE FINAL SYSTEM OBVIOUSLY HAS EXACTLY ONE SOLUTION, NAMELY THE TRIPLE $(11, -4, 3)$.

(CHECK SOLUTION)

GEOMETRICALLY THE THREE EQUATIONS EACH REPRESENT A PLANE. THE SOLUTION (11, -4, 3) IS THEN THE POINT IN SPACE COMMON TO ALL THREE PLANES, I.E. THEIR COMMON INTERSECTION.



THERE ARE MANY WAYS THAT THREE PLANES CAN INTERSECT.



Ex.

$$\begin{cases} x + 2y + 3z = 1 \\ 3x + 2y + z = 1 & -3\cdot\mathrm{I} + \mathrm{II} \\ 7x + 2y - 3z = 1 & -7\cdot\mathrm{I} + \mathrm{III} \end{cases}$$

$$\begin{cases} x + 2y + 3z = 1 \\ -4y - 8z = -2 & \mathrm{II} \div (-\ ) \\ -12y - 24z = -6 & \mathrm{III} \div (-6) \end{cases}$$

$$\begin{cases} x + 2y + 3z = 1 \\ 2y + 4z = 1 \\ \cancel{2y + 4z = 1} & \text{Eliminate III} \end{cases}$$

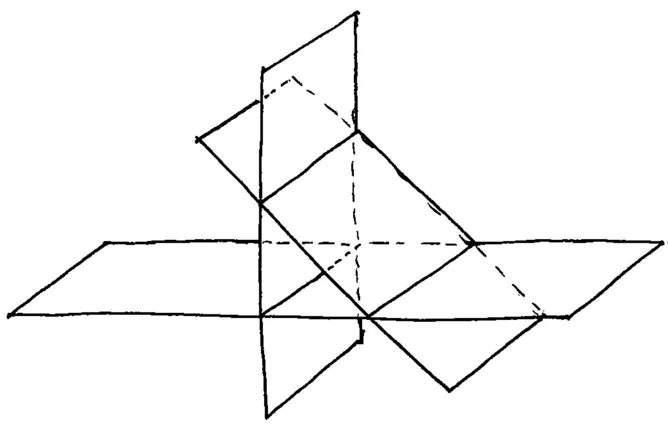
$$\begin{cases} x + 2y + 3z = 1 \\ y + 2z = \frac{1}{2} & \mathrm{II} \div 2 \end{cases}$$

$$\begin{cases} x - z = 0 & -2\cdot\mathrm{II} + \mathrm{I} \\ y + 2z = \frac{1}{2} \end{cases}$$

OBSERVE z CAN BE CHOSEN ARBITRARILY, THEN x AND y ARE UNIQUELY DETERMINED. LET $z = t$, THEN $y = \frac{1}{2} - 2t$, $x = t$.

SOLUTIONS: $\left(t, \frac{1}{2} - 2t, t\right)$ WHERE t IS ANY REAL NUMBER. THIS SET OF POINTS FORMS A LINE IN SPACE.

IT IS ALSO POSSIBLE THAT THE THREE PLANES DO NOT INTERSECT



Ex.

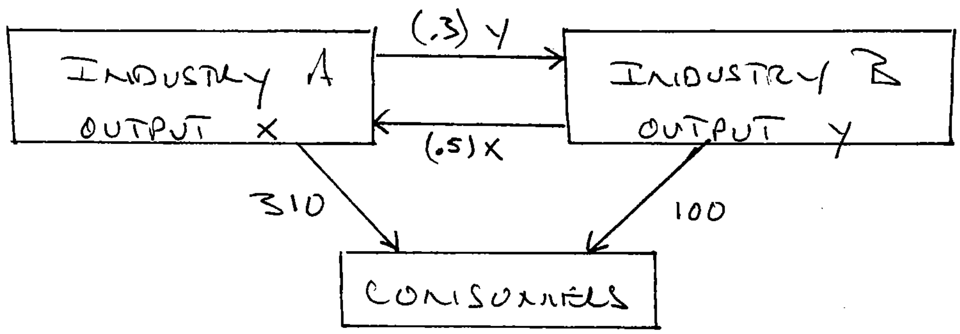
$$\begin{cases} x + 2y + 3z = 1 & \\ x + 3y + 4z = 3 & -1 \cdot \mathrm{I} + \mathrm{II} \\ x + 4y + 5z = 4 & -1 \cdot \mathrm{I} + \mathrm{III} \end{cases}$$

$$\begin{cases} x + 2y + 3z = 1 & -2 \cdot \mathrm{II} + \mathrm{I} \\ y + z = 2 & \\ 2y + 2z = 3 & -2 \cdot \mathrm{II} + \mathrm{III} \end{cases}$$

$$\begin{cases} x + z = -1 \\ y + z = 2 \\ 0 = -1 \end{cases}$$

NO MATTER WHAT VALUES ARE SUBSTITUTED FOR (x, y, z) THE EQUATION $0 = -1$ WILL ALWAYS BE FALSE. $\therefore$ THIS SYSTEM HAS NO SOLUTION.

Ex. Leontief Input-Output Model



Problem: Determine outputs x and y.

$$\begin{cases} x = (.3)y + 310 \\ y = (.5)x + 100 \end{cases}$$

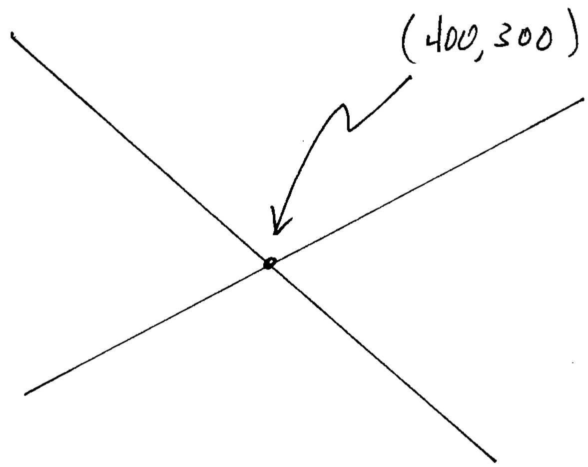
$$\begin{cases} 10x - 3y = 3100 \\ -5x + 10y = 1000 \end{cases}$$

$$\begin{cases} x - 2y = -200 \\ 10x - 3y = 3100 \end{cases}$$

$$\begin{cases} x - 2y = -200 \\ 17y = 5100 \end{cases}$$

$$\begin{cases} x - 2y = -200 \\ y = 300 \end{cases}$$

$$\begin{cases} x = 400 \\ y = 300 \end{cases}$$



Homework

Read 1.1, 1.2

Do (1.1) 2-14 even, 18, 26, 30, 42