8.1.2 AUTONOMOUS EQUATIONS

$$\frac{dy}{dx} = g(y)$$

THESE ARE EQUATIONS IN WHICH THE RIGHT HAND SIDE DOES NOT DEPEND EXPLICITLY ON 'TIME' x.

WE FORMALLY SEPARATE THE VARIABLES x AND y

$$\frac{1}{g(y)}\,dy = dx$$

THEN INTEGRATE BOTH SIDES (W.R.T. x ON THE RIGHT, AND W.R.T. y ON THE LEFT.)

$$\int \frac{1}{g(y)}\,dy = \int dx = x + C$$

HOPEFULLY THE L.H.S. WILL BE INTEGRABLE. SUPPOSE H(y) IS AN ANTIDERIVATIVE OF $\frac{1}{g(y)}$. THEN

$$H(y) = x + C$$

IF WE'RE LUCKY AGAIN THE FUNCTION $H(y)$ WILL BE INVERTIBLE, AND

$$y = H^{-1}(x + C)$$

INITIAL VALUES CAN THEN BE USED TO DETERMINE C.

<u>CASE 1</u>: $g(y) = k(y-a)$

HERE k AND a ARE CONSTANTS. WE HAVE

$$\frac{dy}{dx} = k(y-a)$$

$$\int \frac{1}{y-a}\,dy = \int k\,dx$$

$$\ln|y-a| = kx + c_1$$

$$|y-a| = e^{kx+c_1} = e^{c_1} \cdot e^{kx}$$

$$y - a = \underbrace{\left(\pm e^{c_1}\right)}_{C} \cdot e^{kx}$$

$$\therefore \boxed{y = Ce^{kx} + a}$$

CHECK:

$$\text{LHS} = \frac{dy}{dx} = cKe^{Kx}$$

$$\text{RHS} = K(y-a) = cKe^{Kx}$$

$\therefore$ LHS = RHS FOR All $x \in \mathbb{R}$.

Ex. $$\frac{dy}{dx} = 3 - 5y = -5\left(y - \frac{3}{5}\right)$$

$$\int \frac{1}{y - \frac{3}{5}}\,dy = -5\int dx$$

$$\ln\left|y - \frac{3}{5}\right| = -5x + C_1$$

$$y - \frac{3}{5} = \pm e^{C_1} \cdot e^{-5x}$$

$$\therefore \quad y = ce^{-5x} + \frac{3}{5}$$

Suppose $y(0) = \frac{13}{5}$. Then

$$\frac{13}{5} = y(0) = c \cdot 1 + \frac{3}{5} \Rightarrow c = \frac{10}{5} = 2$$

$$\therefore \quad \boxed{y = 2e^{-5x} + \frac{3}{5}}$$

Ex. $\frac{dh}{ds} = 2h + 1$, $h(0) = 4$

soln: $h(s) = \frac{9e^{2s} - 1}{2}$

Ex. EXPONENTIAL GROWTH

$$\begin{cases} \frac{dN}{dt} = rN \\ N(0) = N_0 \end{cases}$$

soln: $N(t) = N_0 e^{rt}$

THIS IVP MODELS MANY PHENOMENA SUCH AS: POPULATION GROWTH, RADIOACTIVE DECAY, INTEREST, INFLATION.

r IS CALLED THE GROWTH/DECAY RATE.

Ex. $\frac{dN}{dt} = (.3)N$, $N_0 = 20$

$N(t) = 20 e^{(.3)t}$

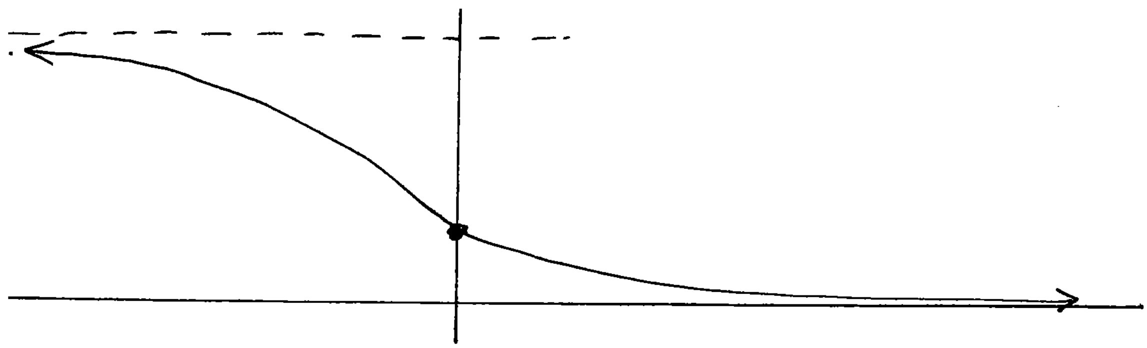
FIND THE POPULATION AT TIME $t = 5$.

$N(5) = 89.633$

CASE 2: $g(y) = K(y-a)(y-b)$

Ex. $\frac{dy}{dx} = y(y-5) \qquad y(0) = 1$

$$y(x) = \frac{5}{1+4e^{5x}}$$



Ex. The logistic equation

$$\begin{cases} \frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \\ N(0) = N_0 \end{cases}$$

$$\frac{dN}{dt} = -\frac{r}{K} N(N-K)$$

$$\int \frac{1}{N(N-K)}\, dN = -\frac{r}{K}\int dt$$

$$\frac{1}{N(N-K)} = \frac{A}{N} + \frac{B}{N-K} \Rightarrow A = -\frac{1}{K},\ B = \frac{1}{K}$$

$$\frac{1}{K}\int\left(\frac{1}{N-K} - \frac{1}{N}\right)dN = -\frac{r}{K}\int dt$$

$$\ln|N-K| - \ln|N| = -rt + C_1$$

$$\frac{N-K}{N} = c\,e^{-rt}$$

$$\therefore \quad c = \frac{N_0 - K}{N_0}$$

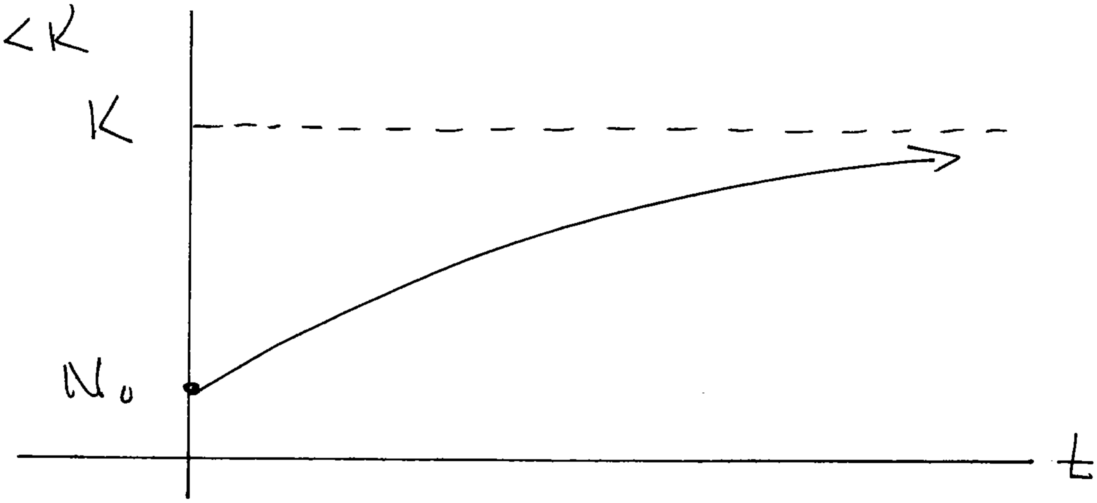
$$\frac{N-K}{N} = \frac{N_0 - K}{N_0}\,e^{-rt}$$

$$\therefore \quad N(t) = \frac{K}{1 + \left(\frac{K}{N_0} - 1\right)e^{-rt}}$$

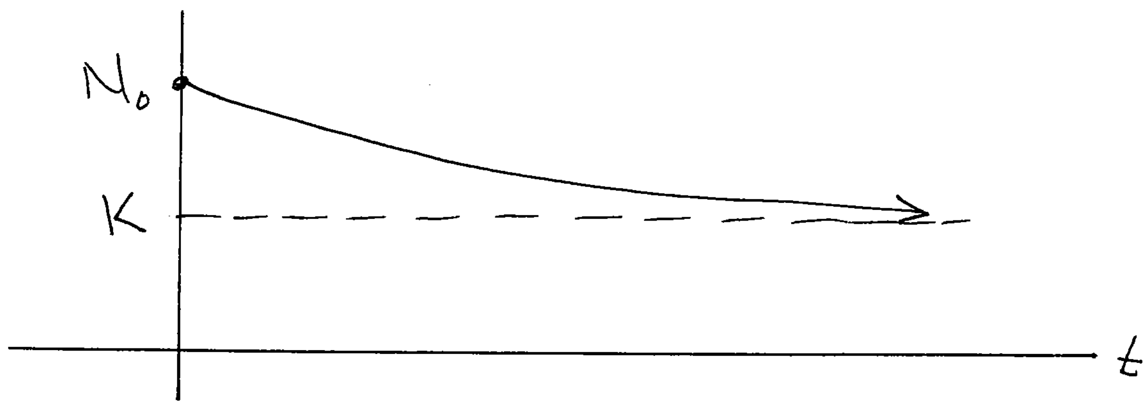
OBSERVE

$$\lim_{t \to \infty} N(t) = K$$

IF $N_0 < K$



OR IF $N_0 > K$



IF $N_0 = K$, THEN $N(t) = K$ IS THE UNIQUE IVP SOLN.

IN GENERAL K REPRESENTS A KIND OF 'SATURATION POINT' FOR THE QUANTITY $N(t)$. K IS ALSO CALLED THE CARRYING CAPACITY

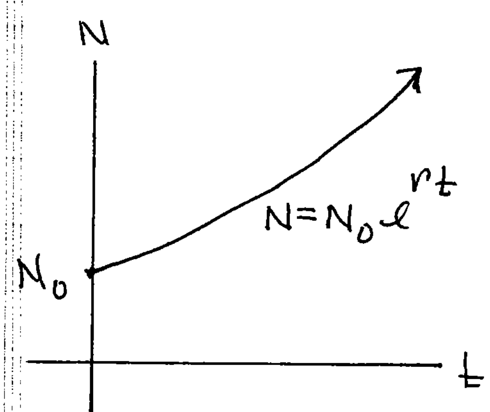
LET'S COMPARE EXPONENTIAL GROWTH TO LOGISTIC GROWTH. WE ASSUME IN BOTH CASES THAT $r > 0$.

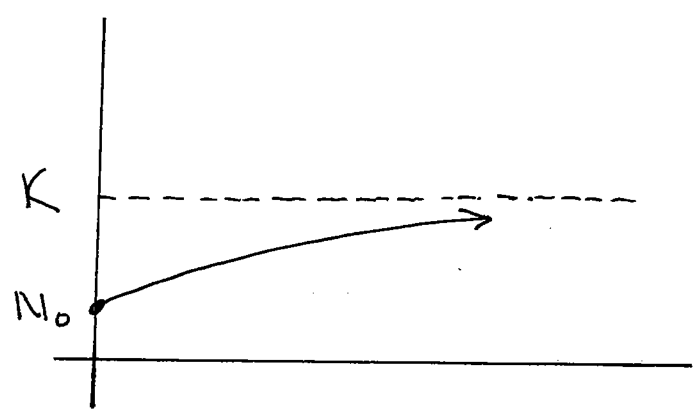
EXPONENTIAL	LOGISTIC
$\frac{dN}{dt} = rN$	$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$
$\frac{1}{N}\frac{dN}{dt} = r$	$\frac{1}{N}\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)$

PER CAPITA GROWTH RATE

CONSTANT

1ST DEGREE POLYNOMIAL

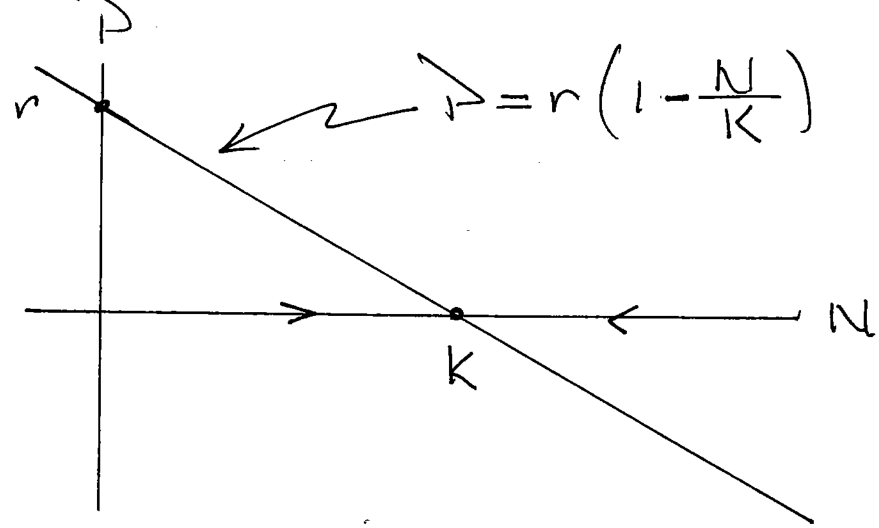




IN THE CASE OF EXPONENTIAL GROWTH THE PER-CAPITA GROWTH RATE

$$\frac{1}{N}\frac{dN}{dt}$$

IS A CONSTANT, WHILE IN THE CASE OF LOGISTIC GROWTH IT IS A 1ST DEGREE POLYNOMIAL:



WHEN THE PER CAPITA GROWTH RATE P IS POSITIVE, THE 'POPULATION' N MUST NECESSARILY INCREASE, WHILE WHEN P IS NEGATIVE N DECREASES.