Chapter 8 Differential Equations

The general first order ordinary differential equation (ODE) is

\[ \frac{dy}{dx} = F(x, y) \]

A solution to this equation on an interval \( I \subseteq \mathbb{R} \) is a function \( y = y(x) \) satisfying

\[ y'(x) = F(x, y(x)) \quad \text{for all } x \in I \]

There is a large and well-developed theory concerning such problems. We restrict our attention to so-called separable equations:

\[ \frac{dy}{dx} = f(x) \cdot g(y) \]

And special cases

\[ \frac{dy}{dx} = f(x) \quad \text{(i.e., } g(y) = 1) \]

\[ \frac{dy}{dx} = g(y) \quad \text{(i.e., } f(x) = 1) \]
Often we will take the independent variable \( x \) to be time, so write \( t \) instead of \( x \). Also, often the dependent variable will be something other than \( y \), and have something like \( N = N(t) \), i.e., problems like

\[
\frac{dN}{dt} = f(t) \quad \text{or} \quad \frac{dN}{dt} = g(N)
\]

(8.1.1) Pure Time ODEs are

\[
\frac{dt}{dx} = f(x) \quad \text{for} \quad x \in I
\]

Notice this is just an integration problem, i.e.,

\[
y(x) = \int_{x_0}^{x} f(u) \, du + C
\]

where \( x_0 \in I \). This follows from the fundamental theorem of calculus (first version).
Observe that
\[
\gamma_0 = \gamma(x_0) = 0 + c = c.
\]

Thus
\[
\gamma(x) = \gamma_0 + \int_{x_0}^{x} f(u) \, du
\]
is the unique solution to the initial value problem (IVP)
\[
\begin{cases}
\frac{dy}{dx} = f(x) & x \in I \\
\gamma(x_0) = \gamma_0
\end{cases}
\]

Note that an ODE like \( \frac{dy}{dx} = f(x) \) in general has infinitely many solutions. A single solution is specified only when the initial value \( \gamma(x_0) \) is given.
Ex. Solve the IVP.

\[ \frac{dv}{dt} = \sin t \]
\[ v(0) = 3 \]

**Solution:** \( v(t) = 4 - \cos t \)

Ex. Solve the IVP.

\[ \frac{dW}{dr} = \ln r \]
\[ W(1) = 4 \]

**Solution:** \( W(r) = r \ln r - r + 5 \)