7.7.1 Taylor Polynomials

All the operations of calculus are easy, even trivial when applied to polynomials

\[ a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \]

For \( k \in \mathbb{Z} \) non-negative we have

\[ \frac{d}{dx} \left[ x^k \right] = k x^{k-1}, \quad \int x^k dx = \frac{x^{k+1}}{k+1} + C \]

One might wish that all functions were polynomials.

The Taylor Polynomials \( P_n(x) \) given functions \( f(x) \) are an attempt to make this wish come true, at least in an approximate sense.

Suppose we wish to approximate a continuous function \( f(x) \) near \( x = 0 \) by a linear (i.e., 1st degree) polynomial \( L(x) \).
Thus the graph $y = L(x)$ will be a line. A reasonable choice for this line is the tangent line to $y = f(x)$ at the point $(0, f(0))$.

![Graph showing tangent line]

\[ y = L(x) \]
\[ y = f(x) \]
\[ (0, f(0)) \]
\[ \text{near } x = 0 \]

\[ L(x) = f(a) + f'(a)(x-a) \]

Note: Generally, we can approximate $f(x)$ near any $x = a$ by the tangent line.

\[ y = L(x) \]
\[ y = f(x) \]
\[ \text{at } a \]
These pictures suggest that the error incurred by approximating \( f(x) \) by \( L(x) \) is small when \( x \) is near \( a \) (or near 0, in the first case).

Ex. Find a linear approximation to \( f(x) = \frac{1}{1-x} \), for \( x \) near 0.

- \( f(0) = 1 \)
- \( f'(0) = 1 \)

\[
\begin{align*}
    f'(x) &= \frac{1}{(1-x)^2}, \\
    L(x) &= 1 + x
\end{align*}
\]

\( L(0) = f(0) \)
\( L'(0) = f'(0) \)

Observe that the linear approximation to \( f(x) \) at \( x = 0 \) satisfied.
We restrict our attention to approximations near \( x = 0 \).

(To obtain approximations near \( x = a \), replace 0 by \( a \), and \( x \) by \( x-a \).)

We seek higher order approximations by requiring that a polynomial of degree \( n \)

\[ P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \]

satisfy

\[ P_n(0) = f(0) \]
\[ P_n'(0) = f'(0) \]
\[ P_n''(0) = f''(0) \]
\[ \vdots \]
\[ P_n^{(k)}(0) = f^{(k)}(0) \]
\[ \vdots \]
\[ P_n^{(n)}(0) = f^{(n)}(0) \]

i.e. we require that the \( k \)-th derivative of \( P_n(x) \) and \( f(x) \) at \( x = 0 \) match for \( k = 0, 1, \ldots, n \).
Observe that there are \( n+1 \) equations in the \( n+1 \) unknown coefficients
\[
a_0, a_1, a_2, \ldots, a_n
\]

Therefore, the approximating polynomial \( P_n(x) \) is defined uniquely.

We call \( P_n(x) \) the Taylor polynomial of degree \( n \) about \( x = 0 \) for \( f(x) \).

Observe:

\[
P_n(x) = a_0 + a_1 x + \cdots + a_n x^n \quad ; \quad P_n(0) = a_0
\]
\[
P'_n(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1} \quad ; \quad P'_n(0) = a_1
\]
\[
P''_n(x) = 2a_2 + 3 \cdot 2a_3 x + \cdots + n(n-1) a_n x^{n-2} \quad ; \quad P''_n(0) = 2a_2
\]
\[
P'''_n(x) = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4 x + \cdots + n(n-1)(n-2) a_n x^{n-3}
\]
\[
\quad ; \quad P'''_n(0) = 3 \cdot 2 \cdot a_3
\]
\[ P_n^{(k)}(x) = k(k-1) \cdots 2 \cdot a_k + \cdots + n(n-1) \cdots (n-k+1) a_n x^{n-k} \]

\[ P_n^{(k)}(0) = k! a_k \]

\[ P_n^{(n)}(x) = n(n-1) \cdots 3 \cdot 2 \]

\[ P_n^{(n)}(0) = n! \]

Thus our \( n+1 \) equations become:

\[ \begin{align*}
A_0 &= f(0) \\
A_1 &= f'(0) \\
2 A_2 &= f''(0) \\
3 \cdot 2 A_3 &= f'''(0) \\
&\vdots \\
k! A_k &= f^{(k)}(0) \\
&\vdots \\
n! A_n &= f^{(n)}(0)
\end{align*} \]

\[ A_0 = f(0) \quad \Rightarrow \quad A_0 = f(0) \]

\[ A_1 = f'(0) \quad \Rightarrow \quad A_1 = f'(0) \]

\[ 2 A_2 = f''(0) \quad \Rightarrow \quad A_2 = \frac{f''(0)}{2} \]

\[ 3 \cdot 2 A_3 = f'''(0) \quad \Rightarrow \quad A_3 = \frac{f'''(0)}{3!} \]

\[ \vdots \]

\[ k! A_k = f^{(k)}(0) \quad \Rightarrow \quad A_k = \frac{f^{(k)}(0)}{k!} \]

\[ n! A_n = f^{(n)}(0) \quad \Rightarrow \quad A_n = \frac{f^{(n)}(0)}{n!} \]

Thus \( P_n(x) \) can be written

\[ P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots + \frac{f^{(n)}(0)}{n!} x^n \]
For more succinctly:

\[ P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \]

This formula gives the Taylor polynomial for \( f(x) \) about \( x = 0 \) of degree \( n \).

Example: \( f(x) = \frac{1}{1-x} \), \( n = 2 \)

\[ P_2(x) = 1 + x + x^2 \]

\[ y = \frac{1}{1-x} \quad \text{at} \quad (0,1) \]

\[ y = 1 + x + x^2 \]
Ex. \( f(x) = \cos x, \ n = 3 \)

Ex. \( f(x) = \sqrt{1+x}, \ n = 4 \)

Ex. \( f(x) = e^{-2x}, \ n = 5 \)

Sometimes one can find a Taylor polynomial by substituting into a smaller one. For instance, one shows easily that \( e^x \) has the 5th degree Taylor polynomial at \( x = 0 \):

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \]

Hence,

\[ e^{-2x} \approx 1 - 2x + \frac{4x^2}{2} - \frac{8x^3}{6} + \frac{16x^4}{24} - \frac{32x^5}{120} \]

\[ = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 - \frac{4}{15}x^5 \]

is the 5th degree Taylor polynomial for \( e^{-2x} \) at \( x = 0 \).
Some well known Taylor Polynomials at $x=0$ are

$$e^x \approx P_n(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k$$

$$\cos x \approx P_{2n}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k}$$

$$\sin x \approx P_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$\ln(1+x) \approx P_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} x^k$$

**Exercise:** Verify these formulas.

See links to Taylor Polynomial Animations on webpage.

**Exercise:** Find Taylor Polynomials for $x=0$ for $e^{-x}$ and $\int_0^x e^t \, dt$ by substituting into the one for $e^x$.  

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