7.4.3 Comparison of Improper Integrals

Sometimes it is enough to know whether or not a given improper integral converges, and its specific value is not needed.

Convergence or divergence can often be proven by comparing the given improper integral to one that is known to either converge or diverge.

Suppose \( 0 \leq f(x) \leq g(x) \) for all \( x \in [a, \infty) \), and suppose it is known that

\[
\int_a^\infty g(x) \, dx
\]

converges, while an antiderivative of \( f(x) \) is not easily found, so we cannot determine the convergence of

\[
\int_a^\infty f(x) \, dx
\]

directly.
Then by Properties (5) and (7) from Section 6.1.3 (pp. 349-352) we have

\[
0 \leq \int_a^z t(x) \, dx \leq \int_a^z g(x) \, dx \leq \int_a^\infty g(x) \, dx
\]

for all \( z \geq a \).

Since \( \int_a^z t(x) \, dx \) is an increasing function of \( z \) and is bounded above by \( \int_a^\infty g(x) \, dx \), we have that
\[ \int_{0}^{\infty} f(x) \, dx = \lim_{z \to \infty} \int_{0}^{z} f(t) \, dt \text{ exists and is finite}, \text{ whence } \int_{0}^{\infty} f(x) \, dx \text{ converge}.

Similarly, if \[ \int_{a}^{\infty} f(x) \, dx \text{ is known to diverge}, while \[ g(x) \text{ is not easily integrable}, we get from the same inequality:

\[ 0 \leq \int_{a}^{z} f(x) \, dx \leq \int_{a}^{z} g(x) \, dx \]

Thus \[ \int_{a}^{\infty} g(x) \, dx \text{ must diverge}.

Ex. Does \[ \int_{0}^{\infty} e^{-x^2} \, dx \text{ converge or diverge?} \text{ Note there is no simple way to write the antiderivative of } e^{-x^2} \]
First split into two terms
\[ \int_{0}^{\infty} e^{-x^2} \, dx = \int_{0}^{1} e^{-x^2} \, dx + \int_{1}^{\infty} e^{-x^2} \, dx \]

The first term is clearly finite.
So it remains to determine the convergence or divergence of
\[ \int_{1}^{\infty} e^{-x^2} \, dx \]

But observe:

\[ x \geq 1 \Rightarrow x^2 \geq x \]
\[ \Rightarrow -x^2 \leq -x \]
\[ \Rightarrow e^{-x^2} \leq e^{-x} \]

Now
\[ \int_{1}^{\infty} e^{-x} \, dx = \lim_{z \to \infty} \left( e^{-1} - e^{-z} \right) = \frac{1}{e} \]

Hence \( \int_{1}^{\infty} e^{-x^2} \, dx \) converges, and

therefore \( \int_{0}^{\infty} e^{-x^2} \, dx \) also converges.

\[ \int_{0}^{\infty} e^{-x^2} \, dx \text{ converges.} \]
In fact one can show by more advanced means that
\[
\int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}.
\]

Ex. Show \( \int_1^\infty \frac{1}{\sqrt{1+x^2}} \, dx \) diverges.

\[\begin{align*}
x = 1 & \implies x^2 \geq 1 > \frac{1}{2} \implies 3x^2 > 1 \\
& \implies 1+x^2 < 4x^2 \\
& \implies \sqrt{1+x^2} < 2x \\
& \implies 0 < \frac{1}{2x} < \frac{1}{\sqrt{1+x^2}}
\end{align*}\]

And \( \int_1^\infty \frac{1}{2x} \, dx = \lim_{z \to \infty} \frac{1}{2} \left( \ln z - \ln 1 \right) = \infty \)

\[\therefore \int_1^\infty \frac{1}{2x} \, dx \text{ diverges } \implies \int_1^\infty \frac{1}{\sqrt{1+x^2}} \, dx \text{ diverges}.
\]

HW 7

(7.3.3) P. 424: 24ab, 26-52 even

(7.4.4) P. 442: 2-26 even, 30, 36ab, 38ab