6.3.5 Rectification of Curves

Consider the problem of finding the length of a curve \( y = f(x) \) from \( x = a \) to \( x = b \).

We approximate the curve by a sequence of straight line segments, then add up the lengths of these segments. If the segments are sufficiently small, this sum serves as a good approximation to the length \( L \).
Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ into $n$ subintervals $a = x_0 < x_1 < \ldots < x_n = b$ and let

$$y_i = f(x_i) \text{ for } 0 \leq i \leq n$$

Obtain a sequence

$$y_0, y_1, \ldots, y_n$$

of points on the $y$-axis. Define

$$\Delta x_i = x_i - x_{i-1} \text{ and } \Delta y_i = y_i - y_{i-1}$$

for $1 \leq i \leq n$.

The $i$th line segment joins points $(x_{i-1}, y_{i-1})$ and $(x_i, y_i)$ and has length

$$\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

by the Pythagorean theorem.
The line $L$ is approximated by the sum

$$L_p = \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$$= \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \cdot \Delta x_i$$

Now $\frac{\Delta y_i}{\Delta x_i}$ is the slope of the $i$th line segment. The MVT says there exists a $c_i \in [x_{i-1}, x_i]$ such that

$$f'(c_i) = \frac{\Delta y_i}{\Delta x_i}$$
Thus

\[ L_p = \sum_{i=1}^{n} \sqrt{1 + (f'(c_i))^2} \Delta x_i \]

This approximation gets better as the partition gets finer. Taking the limit as \(n\) approaches infinity, we have

\[ L = \int_{a}^{b} \sqrt{1 + (f'(x))^2} \, dx \]

Sometimes this is written as

\[ L = \int_{a}^{b} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]

or as

\[ L = \int_{a}^{b} \sqrt{dx^2 + dy^2} \]

The symbol \( \sqrt{dx^2 + dy^2} \) is called the arc length differential and is sometimes denoted \( ds \).
Ex. Find the length of the curve $y^2 = x^3$ from (1,1) to (4,8).

$y = x^{3/2} \Rightarrow \frac{dy}{dx} = \frac{3}{2} x^{1/2}$.

$L = \int_{1}^{4} \sqrt{1 + \left(\frac{3}{2} x^{1/2}\right)^2} \, dx = \int_{1}^{4} \sqrt{1 + \frac{9}{4} x} \, dx$.

Observe
\[
\frac{d}{dx} \left[ \frac{4}{9} \cdot \frac{2}{3} \left( 1 + \frac{9}{4} x \right)^{3/2} \right]
= \frac{4}{9} \cdot \frac{2}{3} \cdot \frac{3}{2} \left( 1 + \frac{9}{4} x \right)^{1/2} \cdot \frac{9}{4}
= \sqrt{1 + \frac{9}{4} x}
\]

Thus
\[ L = \frac{8}{27} \left(1 + \frac{q}{4} x \right)^{3/2} \left[ 1 - \left(\frac{1}{4} \right)^{3/2} \right] \]

4w3 6.3.6 (P. 342)

2-16 even
18, 22, 24, 26, 28, 32
54, 56, 58, 60, 62