6.3.2 Cumulative (or net) Change

Suppose a particle moves along a straight line having position \( s(t) \) at time \( t \)

\[
\begin{array}{cccc}
0 & S_0 & = & S(0) \\
& & & \\
& & \text{distance from origin 0 at time } t.
\end{array}
\]

Recall the velocity (i.e., instantaneous rate of change of position) of the particle is given by

\[ v(t) = \frac{ds}{dt} \]

Suppose we are given the function \( v(t) \) and the initial position \( S_0 \), and we wish to find \( S(t) \).

We must solve the IVP

\[
\begin{cases}
\frac{ds}{dt} = v(t) & \text{for } t > 0 \\
S(0) = S_0
\end{cases}
\]
To solve this problem we first find the general antiderivative of \( v(t) \):

\[
S(t) = \int_{0}^{t} v(u) \, du + C
\]

Since

\[
S_0 = S(0) = \int_{0}^{0} v(u) \, du + C = C
\]

we have

\[
S(t) = S_0 + \int_{0}^{t} v(u) \, du
\]

i.e.

\[
S(t) - S_0 = \int_{0}^{t} v(u) \, du
\]

The expression on the left is the net change in position of the particle from time 0 to time \( t \).

This quantity is also sometimes called cumulative change.
This gives us a physical interpretation of the definite integral:

\[ \int_{a}^{b} f(x) \, dx \]

If \( F(x) \) is any antiderivative of \( f(x) \), then the definite integral above is the net (or cumulative) change in \( F(x) \) from \( x = a \) to \( x = b \), literally:

\[ F(b) - F(a) \]

This interpretation is just a re-statement of the FTC.
6.3.3 AVERAGE VALUES

Recall that the average of a finite list $Y_1, \ldots, Y_n$ of numbers is simply their sum divided by $n$:

$$\text{AVG} = \frac{Y_1 + \cdots + Y_n}{n} = \frac{\sum_{i=1}^{n} Y_i}{n}$$

Suppose $f(x)$ is continuous on $[a, b]$. What should be meant by the phrase "the average value of $f(x)$ on $[a, b]"."

To approximate the average, partition $[a, b]$ into $n$ equally spaced subintervals

$$P = [x_0, x_1, \ldots, x_n]$$

The width of the $i$th subinterval $[x_{i-1}, x_i]$ is

$$\Delta x_i = \Delta x = \frac{b-a}{n}$$

If $n$ is large, then $\Delta x$ is small,
And since $f(x)$ is continuous, it is approximately constant on $[x_{i-1}, x_i]$. Thus, the average value of $f(x)$ (whatever that may mean) is approximately

$$\bar{f} \approx \frac{1}{n} \sum_{i=1}^{n} f(x_i) = \frac{1}{n} \sum_{i=1}^{n} f(x_i)$$

To approximate $f$ by the constant $f(x_i)$ on this sub-interval.

Since $\frac{1}{n} = \frac{\Delta x}{b-a}$ we have

$$\bar{f} \approx \frac{\Delta x}{b-a} \sum_{i=1}^{n} f(x_i)$$

$$= \frac{1}{b-a} \left( \frac{1}{n} \sum_{i=1}^{n} f(x_i) \Delta x \right)$$
Observe that the sum in parentheses is a Riemann sum for \( f(x) \), and upon taking the limit as \( n \to \infty \) we have

\[
\overline{f} = \frac{1}{b-a} \int_a^b f(x) \, dx.
\]

Define

Let \( f(x) \) be continuous on \([a, b]\).

The average value of \( f \) on \([a, b]\) is

\[
\overline{f} = \frac{1}{b-a} \int_a^b f(x) \, dx.
\]

Ex. Suppose the temperature of a body at time \( t \) is given by

\[
T(t) = 1 - (t-2)^2
\]

Find the average temperature from \( t = 1 \) to \( t = 3 \). Ans \[ \frac{2}{3} \]
One would expect that a quantity cannot remain either strictly above or strictly below its average value.

If that quantity is a continuous function, it must therefore assume its average value at some point.

The (mean value theorem for integrals)

If \( f(x) \) is continuous on \([a, b]\)
then there exists \( c \in [a, b] \)
such that

\[
f(c) = f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) \, dx
\]

i.e.

\[
\frac{b}{a} \int_a^b f(x) \, dx = f(c) \cdot (b-a)
\]

\[
\text{Area under curve } y = f(x) \quad \text{Area of a rectangle with base } [a, b] \quad \text{and height } f(c) = f_{\text{avg}}.
\]
Ex. \( T(t) = 1 - (t-2)^2 \). Find \( c \in [1, 3] \) such that

\[ T(c) = \overline{T} \text{avg} = \frac{2}{3} \]

Two answers:

\[ c_1 = 2 + \frac{1}{\sqrt{3}} = 2.5773 \]

\[ c_2 = 2 - \frac{1}{\sqrt{3}} = 1.4226 \]

Observe that the area under \( T = 1 - (t-2)^2 \) from \( t=1 \) to \( t=3 \) is the same as the area of a rectangle with the same base and height \( \overline{T} \text{avg} = \frac{2}{3} \), namely

\[ \text{area} = (3-1) \cdot \frac{2}{3} = \frac{4}{3} \]
Proof of MT for integrals

Since \( f(x) \) is continuous on \([a,b]\) it attains a maximum \( M \) and minimum \( m \) in \([a,b]\).

\[
\begin{array}{c}
\text{M} \\
\mid \\
\text{m} \\
\end{array}
\]

\[
\begin{array}{c}
a \\
\mid \\
b
\end{array}
\]

So \( m \leq f(x) \leq M \) for all \( x \in [a,b] \)

Therefore

\[
m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)
\]

i.e.

\[
m \leq \frac{1}{b-a} \int_a^b f(x) dx \leq M
\]

Set \( I = \frac{1}{b-a} \int_a^b f(x) dx \), so \( m \leq I \leq M \).

By the Intermediate Value Theorem, \( f(x) \) takes on all values between \( m \) and \( M \), hence \( f(c) = I \) for some \( c \in [a,b] \), which completes the proof. \( \Box \)