6.3.2 Cumulative (or net) change

Suppose a particle moves along a straight line having position $s(t)$ at time t

i.e. $s(t)$ = distance from origin O at time t.

Recall the velocity (i.e. instantaneous rate of change of position) of the particle is given by

$$v(t) = \frac{ds}{dt}$$

Suppose we are given the function $v(t)$ and the initial position s_0, and we wish to find $s(t)$.

We must solve the IVP

$$\begin{cases} \dfrac{ds}{dt} = v(t) & \text{for } t > 0 \\ s(0) = s_0 \end{cases}$$

TO SOLVE THIS PROBLEM WE FIRST FIND THE GENERAL ANTIDERIVATIVE OF $v(t)$:

$$s(t) = \int_0^t v(u)\,du + C$$

SINCE

$$s_0 = s(0) = \int_0^0 v(u)\,du + C = C$$

WE HAVE

$$\boxed{s(t) = s_0 + \int_0^t v(u)\,du}$$

i.e.

$$s(t) - s_0 = \int_0^t v(u)\,du$$

THE EXPRESSION ON THE LEFT IS THE <u>NET CHANGE</u> IN POSITION OF THE PARTICLE FROM TIME 0 TO TIME t.

THIS QUANTITY IS ALSO SOMETIMES CALLED <u>CUMULATIVE CHANGE</u>

THIS GIVES US A PHYSICAL INTERPRETATION OF THE DEFINITE INTEGRAL:

$$\int_a^b f(x)\,dx$$

IF $F(x)$ IS ANY ANTIDERIVATIVE OF $f(x)$ THEN THE DEFINITE INTEGRAL ABOVE IS THE NET (OR CUMULATIVE) CHANGE IN $F(x)$ FROM $x=a$ TO $x=b$, LITERALLY:

$$F(b)-F(a) \quad .$$

THIS INTERPRETATION IS JUST A RE-STATEMENT OF THE FTC.

6.3.3 AVERAGE VALUES

RECALL THAT THE AVERAGE OF A FINITE LIST $Y_1, \ldots, Y_n$ OF NUMBERS IS SIMPLY THEIR SUM DIVIDED BY n

$$AVG = \frac{Y_1 + \cdots + Y_n}{n} = \frac{\sum_{i=1}^{n} Y_i}{n}.$$

SUPPOSE $f(x)$ IS CONTINUOUS ON $[a,b]$. WHAT SHOULD BE MEANT BY THE PHRASE "THE AVERAGE VALUE OF $f(x)$ ON $[a,b]$."

TO APPROXIMATE AN ANSWER, PARTITION $[a,b]$ INTO n EQUALLY SPACED SUBINTERVALS

$$P = [\overset{a}{\overset{\shortparallel}{x_0}}, x_1, \ldots, \overset{b}{\overset{\shortparallel}{x_n}}]$$

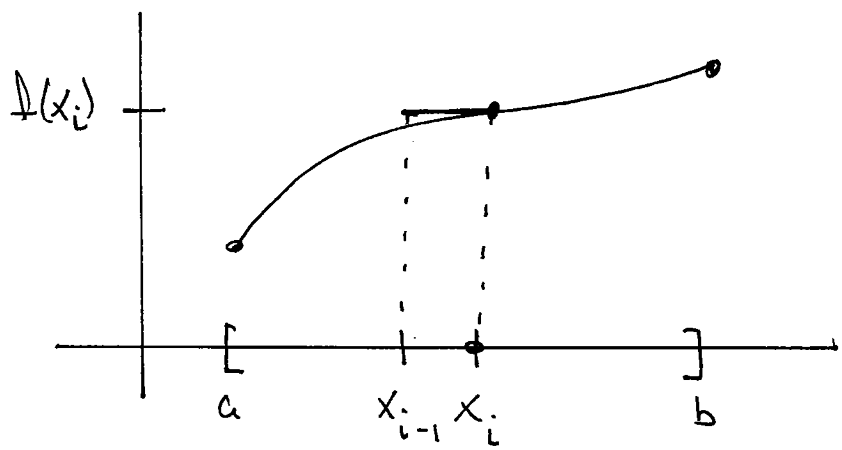
THE WIDTH OF i^{TH} SUBINTERVAL $[x_{i-1}, x_i]$ IS

$$\Delta x_i = \Delta x = \frac{b-a}{n}$$

IF n IS LARGE, THEN Δx IS SMALL,

AND SINCE $f(x)$ IS CONTINUOUS, IT IS APPROXIMATELY CONSTANT ON $[x_{i-1}, x_i]$. THUS, THE AVERAGE VALUE OF $f(x)$ (WHATEVER THAT MAY MEAN) IS APPROXIMATELY

$$\bar{f} \approx \frac{\sum_{i=1}^{n} f(x_i)}{n} = \frac{1}{n} \sum_{i=1}^{n} f(x_i)$$



APPROXIMATE f BY THE CONST. $f(x_i)$ ON THIS SUB-INTERVAL.

SINCE $\frac{1}{n} = \frac{\Delta x}{b-a}$ WE HAVE

$$\bar{f} \approx \frac{\Delta x}{b-a} \sum_{i=1}^{n} f(x_i)$$

$$= \frac{1}{b-a} \left(\sum_{i=1}^{n} f(x_i) \Delta x \right)$$

OBSERVE THAT THE SUM IN PARENTHESIS IS A RIEMANN SUM FOR $f(x)$, AND UPON TAKING THE LIMIT AS $n \to \infty$ WE HAVE

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x)\,dx .$$

DEFN:
LET $f(x)$ BE CONTINUOUS ON $[a,b]$. THE AVERAGE VALUE OF f ON $[a,b]$ IS

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x)\,dx .$$

EX. SUPPOSE THE TEMPERATURE OF A BODY AT TIME t IS GIVEN BY

$$T(t) = 1-(t-2)^2$$

FIND THE AVERAGE TEMPERATURE FROM $t=1$ TO $t=3$. ANS $\boxed{\frac{2}{3}}$

ONE WOULD EXPECT THAT A QUANTITY CANNOT REMAIN EITHER STRICTLY ABOVE OR STRICTLY BELOW ITS AVERAGE VALUE.

IF THAT QUANTITY IS A CONTINUOUS FUNCTION, IT MUST THEREFORE ASSUME ITS AVERAGE VALUE AT SOME POINT.

<u>THM</u> (MEAN VALUE THM FOR INTEGRALS)

IF $f(x)$ IS CONTINUOUS ON $[a,b]$ THEN THERE EXISTS A $c \in [a,b]$ SUCH THAT

$$f(c) = f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

i.e.

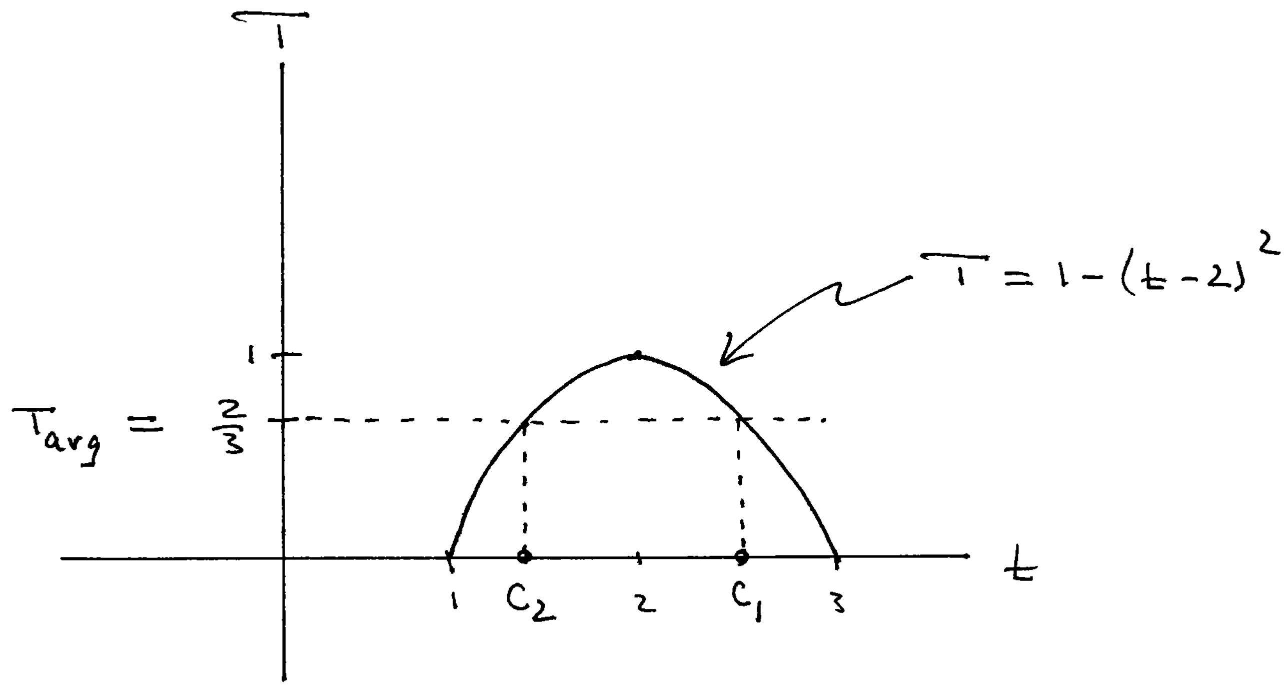
$$\underbrace{\int_a^b f(x)\,dx}_{\text{AREA UNDER CURVE } y=f(x)} = \underbrace{f(c)\cdot(b-a)}_{\text{AREA OF A RECTANGLE WITH BASE } [a,b] \text{ AND HEIGHT } f(c)=f_{avg}}.$$

<u>Ex.</u> $T(t) = 1-(t-2)^2$. FIND $c \in [1,3]$ SUCH THAT

$$T(c) = T_{avg} = \frac{2}{3}$$

TWO ANSWERS: $c_1 = 2 + \frac{1}{\sqrt{3}} = 2.5773$

$$c_2 = 2 - \frac{1}{\sqrt{3}} = 1.4226$$

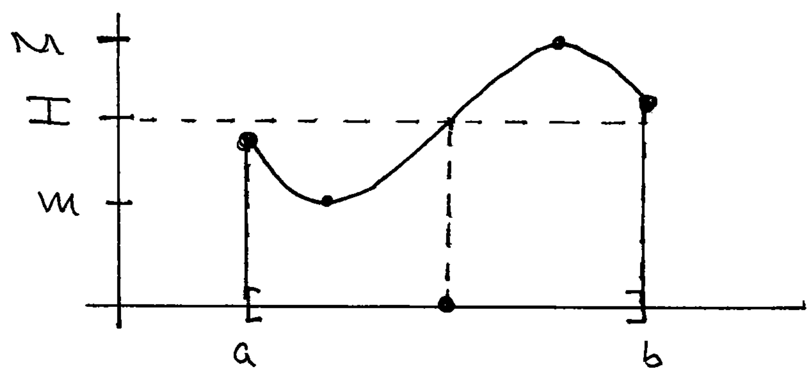


OBSERVE THAT THE AREA UNDER $T = 1-(t-2)^2$ FROM $t=1$ TO $t=3$ IS THE SAME AS THE AREA OF A RECTANGLE WITH THE SAME BASE AND HEIGHT $T_{avg} = \frac{2}{3}$, NAMELY

$$\text{AREA} = (3-1)\cdot\frac{2}{3} = \frac{4}{3}.$$

PROOF OF MVT FOR INTEGRALS

SINCE $f(x)$ IS CONTINUOUS ON $[a,b]$ IT ATTAINS A MAXIMUM M AND MINIMUM m IN $[a,b]$



SO $m \le f(x) \le M$ FOR ALL $x \in [a,b]$

THEREFORE

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

i.e.

$$m \le \frac{1}{b-a} \int_a^b f(x)\,dx \le M$$

SET $I = \frac{1}{b-a} \int_a^b f(x)\,dx$, SO $m \le I \le M$.
BY THE INTERMEDIATE VALUE THEOREM, $f(x)$ TAKES ON ALL VALUES BETWEEN m AND M, HENCE $f(c) = I$ FOR SOME $c \in [a,b]$, WHICH COMPLETES THE PROOF. ///