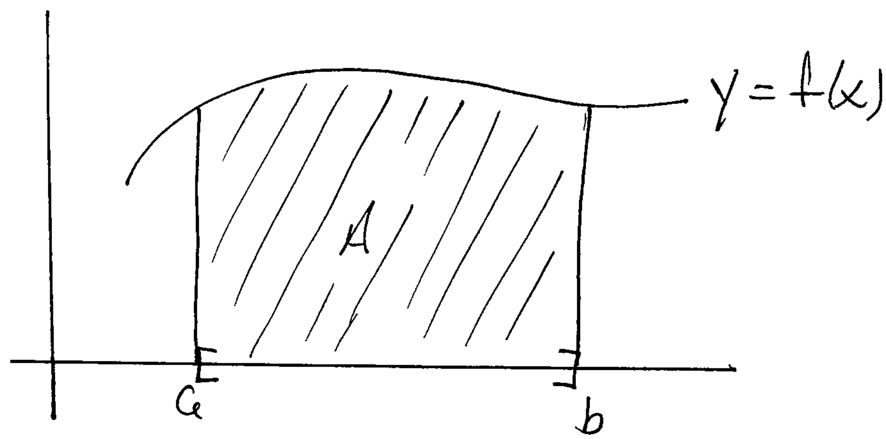
6.3 APPLICATIONS

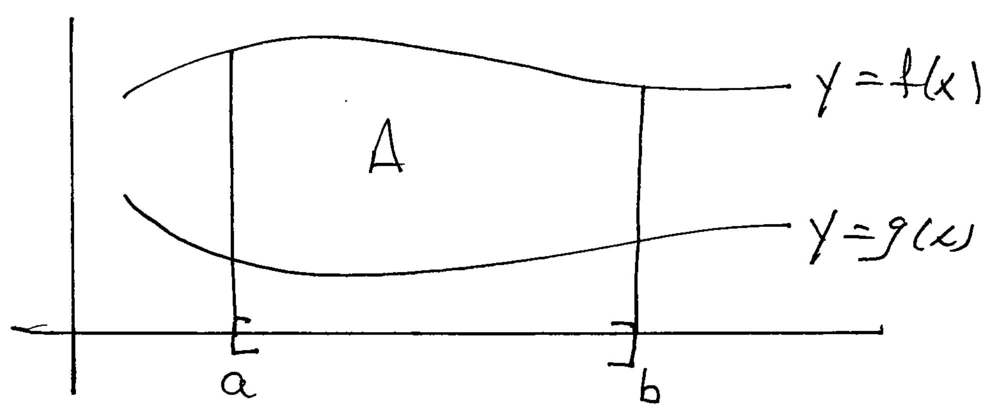
6.3.1 AREA

RECALL THAT THE DEFINITE INTEGRAL WAS DEVELOPED AS A WAY TO FIND THE AREA UNDER A CURVE



$$A = \int_a^b f(x)\,dx$$

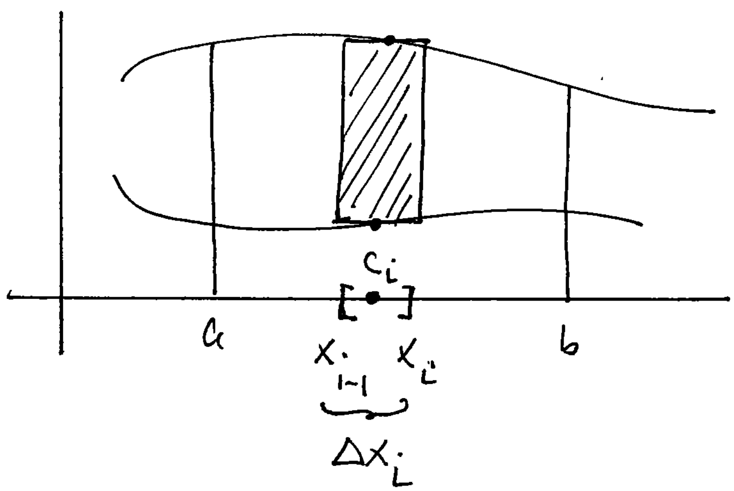
MORE GENERALLY, IF WE WISH TO FIND THE AREA BETWEEN TWO CURVES $y = f(x)$ AND $y = g(x)$ WE CAN SUBTRACT TWO AREAS



$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$= \int_a^b [f(x) - g(x)]\,dx$$

THIS LAST FORMULA CAN ITSELF BE JUSTIFIED USING THE RIEMANN SUM APPROACH.



i.e. THE AREA BETWEEN CURVES IS APPROXIMATED BY

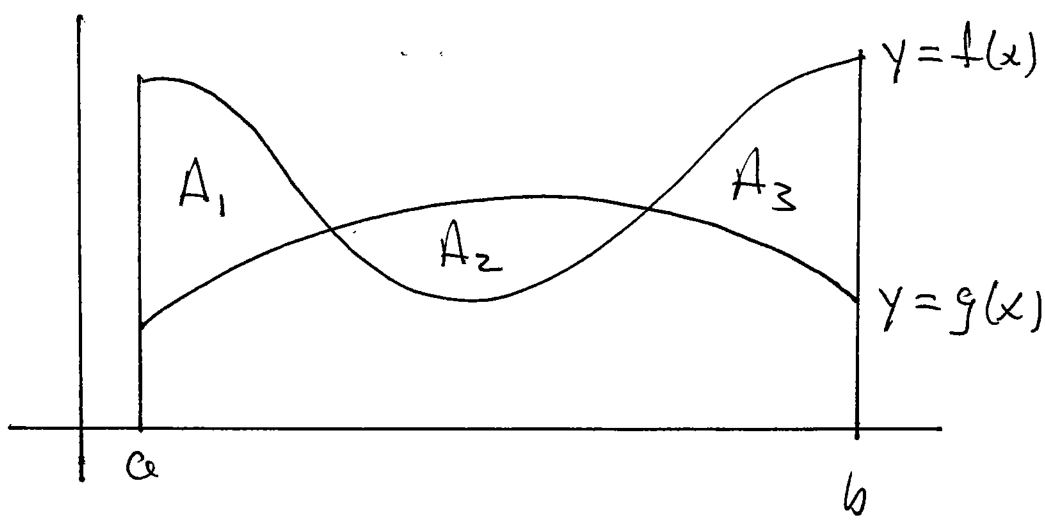
$$\sum_{i=1}^{n} [f(c_i) - g(c_i)]\,\Delta x_i$$

WHERE $P = [x_0, \ldots, x_n]$ IS A PARTITION OF $[a, b]$ AND $c_i \in [x_{i-1}, x_i]$ $(1 \le i \le n)$.

TAKING THE LIMIT OF SUCH SUMS AS $\|P\| \to 0$, WE GET

$$A = \int_a^b [f(x) - g(x)]\,dx .$$

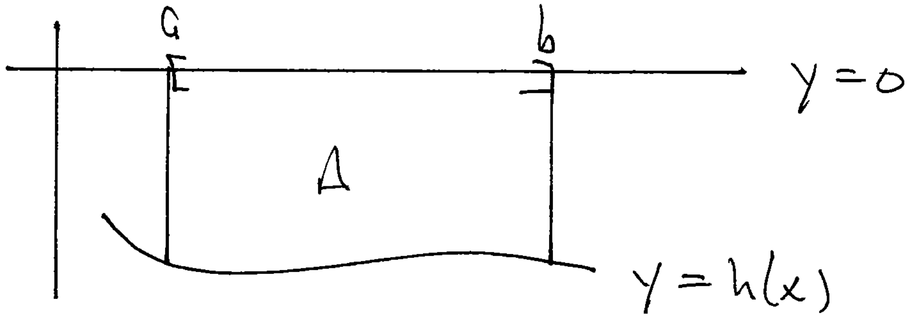
NOTE ALSO THAT THIS IS PROPERLY UNDERSTOOD AS A SIGNED AREA, i.e. POSITIVE WHERE $f(x) > g(x)$ AND NEGATIVE WHERE $f(x) < g(x)$



$$\int_a^b [f(x) - g(x)]\,dx = A_1 - A_2 + A_3 .$$

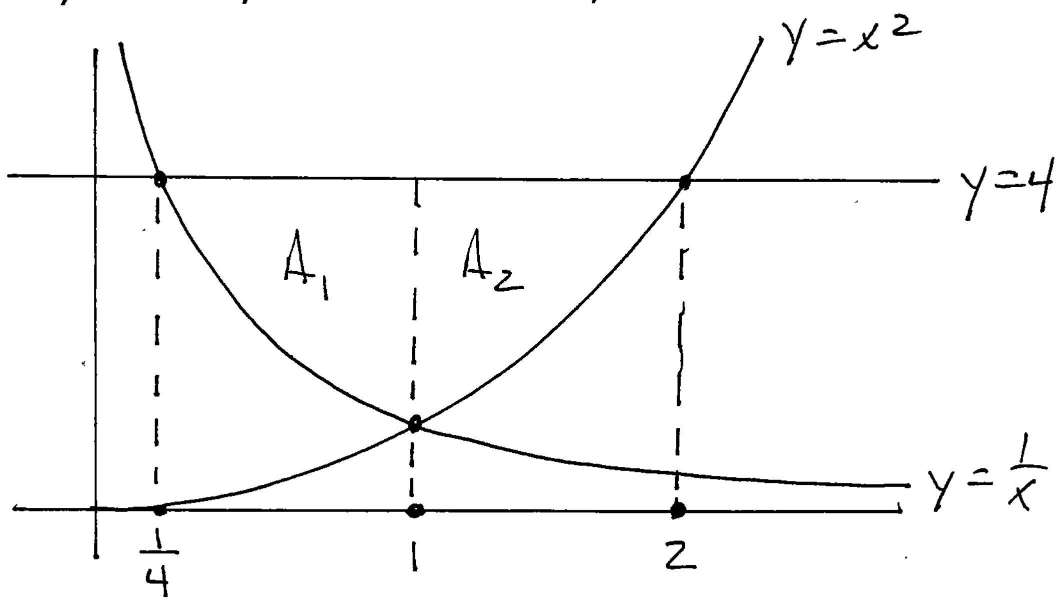
FOR INSTANCE, IF $h(x) \le 0$ FOR ALL $x \in [a, b]$ THEN THE (POSITIVE) AREA BETWEEN $y = h(x)$ AND THE x-AXIS $y = 0$ IS GIVEN BY

$$A = \int_a^b [0 - h(x)]\,dx = -\int_a^b h(x)\,dx$$



EX. FIND AREA BDD BY $y = 1 - x^2$, $y = x^2$. ANS $\boxed{\frac{2\sqrt{2}}{3}}$

EX. FIND AREA IN 1^{ST} QUADRANT BOUNDED BY $y = x^2$, $y = \frac{1}{x}$, AND $y = 4$

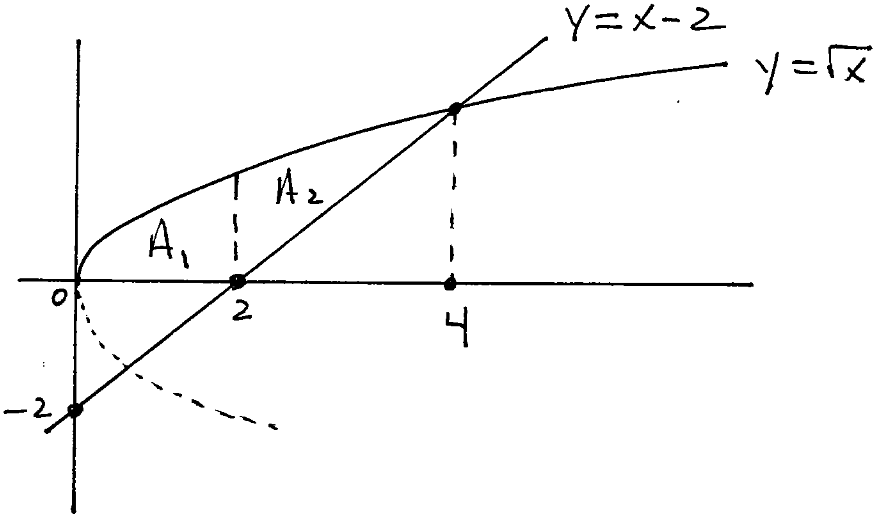


$$A = A_1 + A_2 = \int_{1/4}^{1} \left(4 - \frac{1}{x}\right) dx + \int_1^2 (4 - x^2)\,dx$$

$$= \frac{14}{3} - \ln(4) \approx 3.2803$$

THIS EXAMPLE ILLUSTRATES THE IMPORTANCE OF ALWAYS DRAWING A PICTURE OF THE REGION WHOSE AREA YOU ARE CALCULATING.

<u>EX.</u> FIND THE AREA BOUNDED BY $y=\sqrt{x}$, $y=x-2$, AND $y=0$.

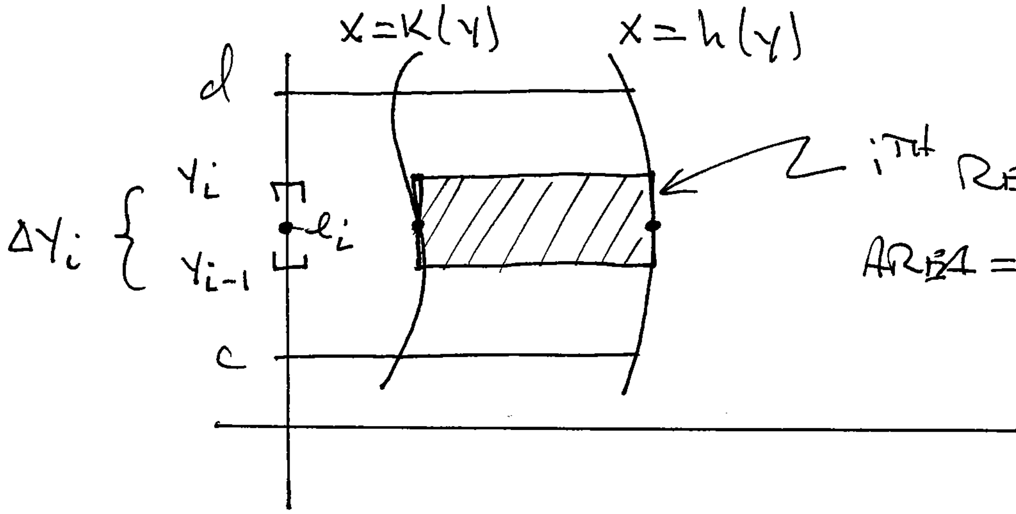


$$A = A_1 + A_2 = \int_0^2 \sqrt{x}\,dx + \int_2^4 \left[\sqrt{x} - (x-2)\right] dx$$

$$= \frac{10}{3}$$

NOTICE THAT IN THE PRECEDING TWO EXAMPLES, IT WAS NECESSARY TO SPLIT THE REGION IN QUESTION INTO TWO SMALLER REGIONS SINCE THE LOWER BOUNDARY WAS COMPOSED OF TWO DIFFERENT CURVES.

OFTEN IN SUCH CASES IT IS POSSIBLE TO SET UP SUCH AN AREA CALCULATION AS THE LIMIT OF RIEMANN SUMS WITH HORIZONTAL RECTANGLES, (RATHER THAN VERTICAL RECTANGLES.)



i^{th} RECTANGLE
AREA $= (h(\ell_i) - k(\ell_i))\Delta y_i$

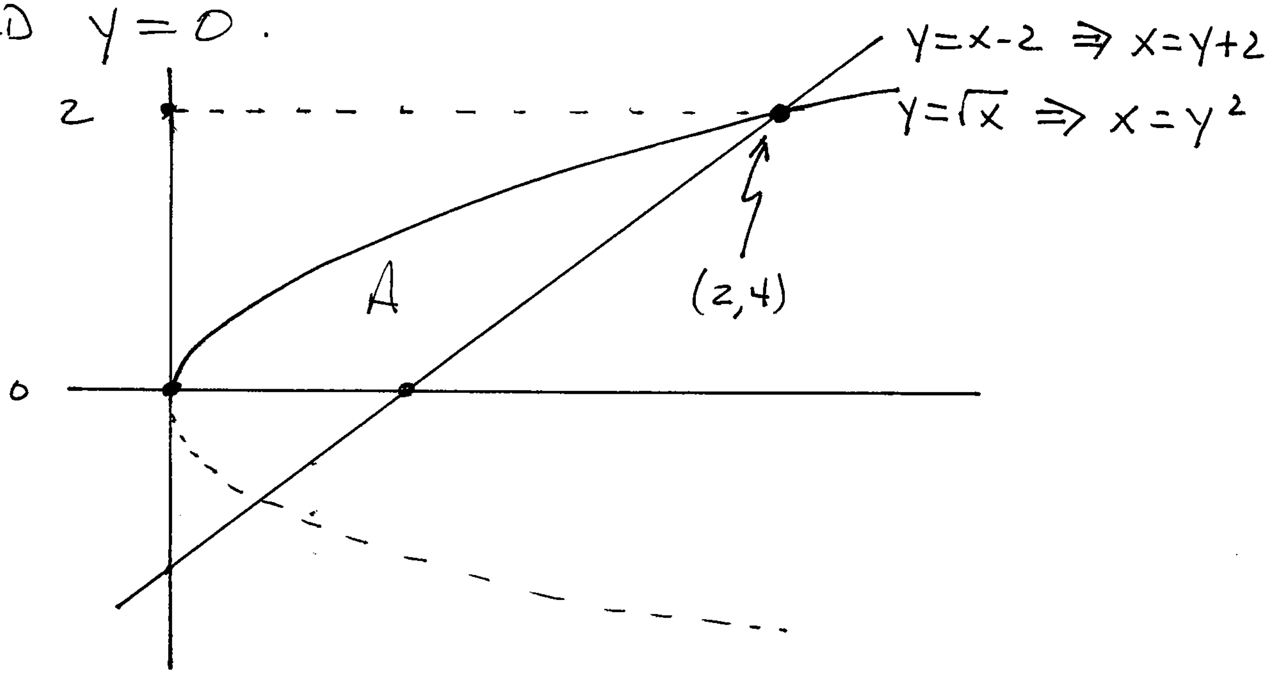
$$A \approx \sum_{i=1}^{n} [h(\ell_i) - k(\ell_i)]\Delta y_i$$

WHERE $P = [y_0, y_1, \ldots, y_n]$ IS A PARTITION OF $[c, d]$ ON THE y-AXIS AND $\ell_i \in [y_{i-1}, y_i]$ FOR $1 \le i \le n$.

TAKING THE LIMIT AS $\|P\| \to 0$ WE GET

$$A = \int_c^d [h(y) - k(y)]\, dy .$$

EX. FIND AREA BETWEEN $y=\sqrt{x}$, $y=x-2$, AND $y=0$.



$$A = \int_0^2 \left[(y+2) - y^2\right] dy = \ldots = \frac{10}{3}$$

EXERCISE

RE-DO THE 2^{ND} EXAMPLE IN THIS SECTION, i.e. FIND THE AREA BOUNDED BY $y=x^2$, $y=\frac{1}{x}$, $y=4$ IN THE 1^{ST} QUADRANT.

KEEP IN MIND THAT THE ANSWER MUST BE THE SAME AS BEFORE! $14/3 - \ln(4)$