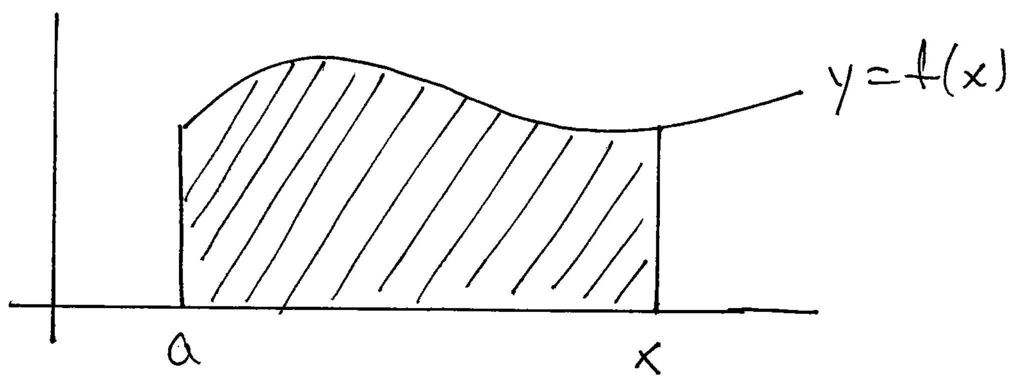
## 6.2.1 THE FUNDAMENTAL THEOREM OF CALCULUS

THE FTC PROVIDES A LINK BETWEEN THE OPERATIONS OF DIFFERENTIATION AND INTEGRATION.

LET $f(x)$ BE CONTINUOUS ON $[a,b]$, AND DEFINE

$$F(x) = \int_a^x f(u)\,du$$




$F(x)$ = SIGNED AREA FROM $a$ TO $x$ UNDER $y = f(x)$

NOW WE CALCULATE $F'(x)$ (USING SOME GEOMETRIC INTUITION).

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left[ F(x+h) - F(x) \right]$$

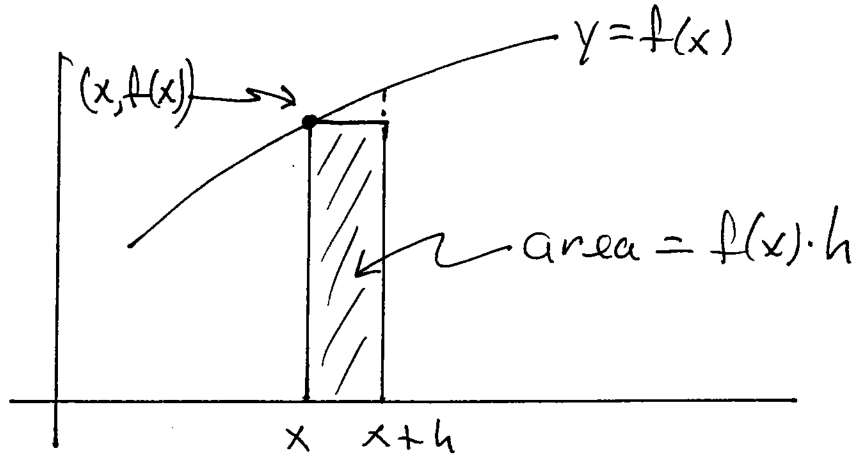
$$= \lim_{h \to 0} \frac{1}{h} \left[ \int_a^{x+h} f(u)\,du - \int_a^{x} f(u)\,du \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[ \int_x^{a} f(u)\,du + \int_a^{x+h} f(u)\,du \right] \quad \left\{ \text{BY PROP (2)} \right.$$

$$= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(u)\,du \quad \left\{ \text{BY PROP (5)} \right.$$

NOW OBSERVE THAT $\int_x^{x+h} f(u)\,du$

IS THE SIGNED AREA UNDER $y = f(x)$ FROM $x$ TO $x+h$, WHICH IS APPROXIMATED BY $f(x)h$



i.e. $\frac{1}{h}\int_x^{x+h} f(u)\,du \approx \frac{1}{h}\cdot f(x)\cdot h = f(x).$

THIS APPROXIMATION GETS BETTER WITH SMALLER $h$, WHENCE

$$F'(x) = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(u)\,du = f(x)$$

i.e. $F(x)$ is AN ANTIDERIVATIVE OF $f(x)$,

STATED ANOTHER WAY: $\frac{d}{dx}\int_a^x f(u)\,du = f(x)$.

<u>THM FTA 1ST VERSION</u>

IF $f(x)$ is CONTINUOUS ON $[a,b]$ THEN THE FUNCTION $F(x)$ DEFINED BY

$$F(x) = \int_a^x f(u)\,du$$

is CONTINUOUS ON $[a,b]$, DIFFERENTIABLE ON $(a,b)$, AND

$$\frac{d}{dx} F(x) = f(x)$$

WE'LL GIVE A MORE RIGOROUS PROOF OF THIS FACT LATER.

EX. COMPUTE THE DERIVATIVE $\left(\frac{d}{dx}\right)$ OF THE FOLLOWING FUNCTIONS OF $x$:

- $\displaystyle\int_0^x \sin(u)\,du$
- $\displaystyle\int_1^x \ln(u)\,du$
- $\displaystyle\int_3^x \frac{1}{1+u^5}\,du$
- $\displaystyle\int_3^x \ln\left(\frac{1}{1+u^5}\right)du$
- $\displaystyle\int_0^x \sin(\sin(\sin(u)))\,du$
- $\displaystyle\int_a^x \mathrm{HORROR}(u)\,du$

Ex. $\frac{d}{dx}\int_x^5 (3u^3-12u+1)\,du$

Ex. $\frac{d}{dx}\int_0^{x^2} \sin(u)\,du = \sin(x^2)\cdot 2x$

Ex. $\frac{d}{dx}\int_0^{\ln(x)} \sin(u)\,du = \sin(\ln(x))\cdot \frac{1}{x}$

Ex. $\frac{d}{dx}\int_0^{\ln(x^2)} \sin(u)\,du = \sin(\ln(x^2))\cdot \frac{1}{x^2}\cdot 2x$

Ex. $\frac{d}{dx}\int_{x^2}^{x^3} \sin(u)\,du$

$$= \frac{d}{dx}\left[\int_{x^2}^{0} \sin(u)\,du + \int_0^{x^3} \sin(u)\,du\right]$$

$$= \frac{d}{dx}\left[\int_0^{x^3} \sin(u)\,du - \int_0^{x^2} \sin(u)\,du\right]$$

$$= \sin(x^3)\cdot 3x^2 - \sin(x^2)\cdot 2x$$

Thm: (LEIBNIZ'S RULE)

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(u)\,du = f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)$$

Ex.

$$\frac{d}{dx}\int_{\ln(x)}^{\sin(x)} \cos(u)\,du$$

Ex

$$\frac{d}{dx}\int_{\ln(x)}^{2x+1} e^{u^2}\,du$$

Thm: (LEIBNIZ'S RULE)

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(u)\,du = f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)$$

Ex.

$$\frac{d}{dx}\int_{\ln(x)}^{\sin(x)} \cos(u)\,du$$

Ex

$$\frac{d}{dx}\int_{\ln(x)}^{2x+1} e^{u^2}\,du$$

## PROOF OF FTC ($1^{ST}$ VERSION)

WE MUST SHOW

$$\frac{d}{dx}\int_a^x f(u)\,du = f(x).$$

AS WE'VE SEEN

$$\frac{d}{dx}\int_a^x f(u)\,du = \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(u)\,du$$

THUS WE MUST PROVE THE LIMIT STATEMENT

$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(u)\,du = f(x)$$

LET $m$ AND $M$ BE THE MINIMUM AND MAXIMUM VALUES (RESPECTIVELY) ACHIEVED BY $f(u)$ ON $[x, x+h]$. i.e.

$$m \le f(u) \le M$$

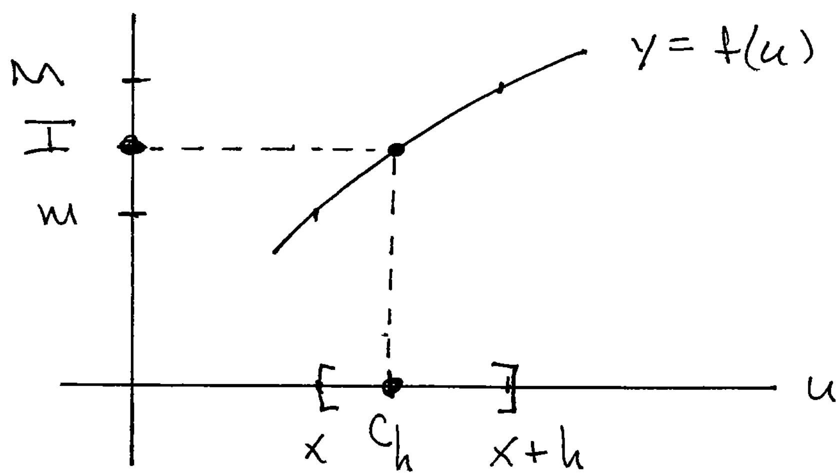
FOR ALL $u \in [x, x+h]$.

BY ONE OF OUR PROPERTIES ON P. 352, WE HAVE

$$mh \le \int_x^{x+h} f(u)\,du \le Mh$$

WHENCE

$$m \le \underbrace{\frac{1}{h}\int_x^{x+h} f(u)\,du}_{I} \le M$$




SINCE $f(u)$ IS CONTINUOUS, THE INTERMEDIATE VALUE THEOREM (P. 149) SAYS THERE EXISTS A NUMBER $c_h \in [x, x+h]$ SUCH THAT

$$f(c_h) = I = \frac{1}{h}\int_x^{x+h} f(u)\,du$$

OBVIOUSLY $c_h \to x$ AS $h \to 0$.
THUS, AGAIN BY THE CONTINUITY
OF $f$:

$$\frac{d}{dx}\int_a^x f(u)\,du = \lim_{h\to 0}\int_x^{x+h} f(u)\,du$$

$$= \lim_{h\to 0} f(c_h)$$

$$= f\left(\lim_{h\to 0} c_h\right)$$

$$= f(x),$$

AS REQUIRED. ///.