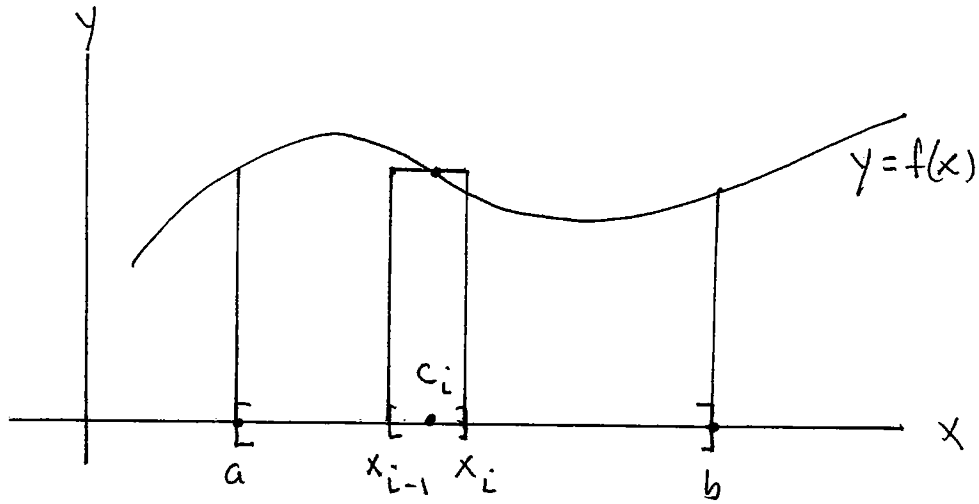
6.1.2 RIEMANN SUMS & INTEGRALS

WE RETURN TO THE MORE GENERAL PROBLEM OF FINDING THE AREA UNDER THE GRAPH $y = f(x)$ lying OVER THE INTERVAL $[a, b]$.



AGAIN WE DIVIDE $[a, b]$ INTO n SUB-INTERVALS

$$[x_0, x_1], [x_1, x_2], \ldots\ldots, [x_{n-1}, x_n]$$

WHERE $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. WE DO NOT INSIST HOWEVER THAT ALL SUBINTERVALS BE OF THE SAME LENGTH. LET

$$\Delta x_i = \text{length}\,[x_{i-1}, x_i] = x_i - x_{i-1},$$

THE LENGTH OF THE i^{TH} SUB-INTERVAL

SUCH A SUBDIVISION IS CALLED A PARTITION OF $[a,b]$ AND IS DENOTED

$$P = [x_0, x_1, x_2, \ldots, x_n]$$

WE DEFINE THE NORM OF P TO BE THE WIDTH OF THE LARGEST SUBINTERVAL

$$\|P\| = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\}$$

OBSERVE THAT $\|P\|$ SERVES AS A MEASURE OF THE 'FINENESS' OF P.

NOW SELECT POINTS

$$c_i \in [x_{i-1}, x_i] \qquad (\text{FOR } 1 \le i \le n)$$

AND ERECT A RECTANGLE OF HEIGHT $f(c_i)$ OVER $[x_{i-1}, x_i]$.

NOTE c_i NEED NO LONGER BE AN END POINT OF $[x_{i-1}, x_i]$.

THE AREA OF THE i^{TH} RECTANGLE IS THEN

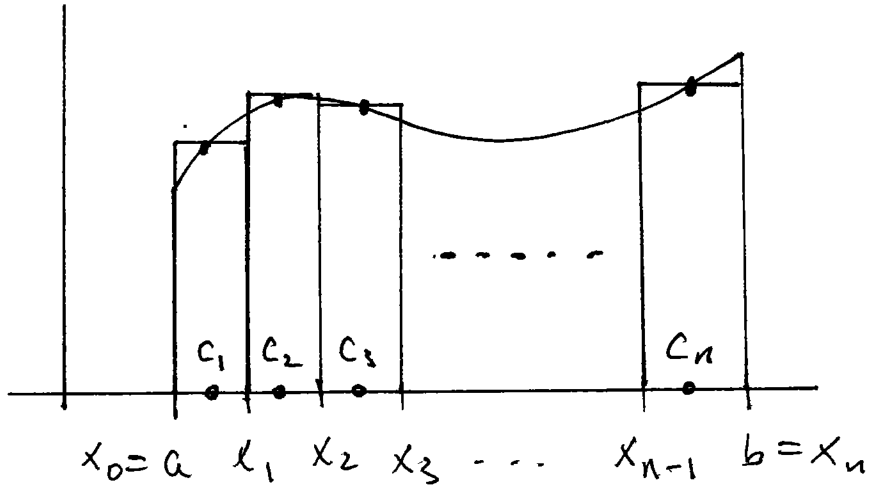
$$f(c_i) \cdot \Delta x_i$$

WE APPROXIMATE THE AREA UNDER $y = f(x)$ FROM a TO b BY THE SUM

$$\boxed{S_P = \sum_{i=1}^{n} f(c_i)\Delta x_i}$$

THE ABOVE SUMMATION IS CALLED A <u>RIEMANN SUM</u> FOR f ON $[a,b]$.

OBSERVE THAT S_P DEPENDS ON THE PARTITION P (AS INDICATED BY THE NOTATION) AND ON THE CHOICE OF $c_i \in [x_{i-1}, x_i]$ $(1 \leq i \leq n)$.



EX

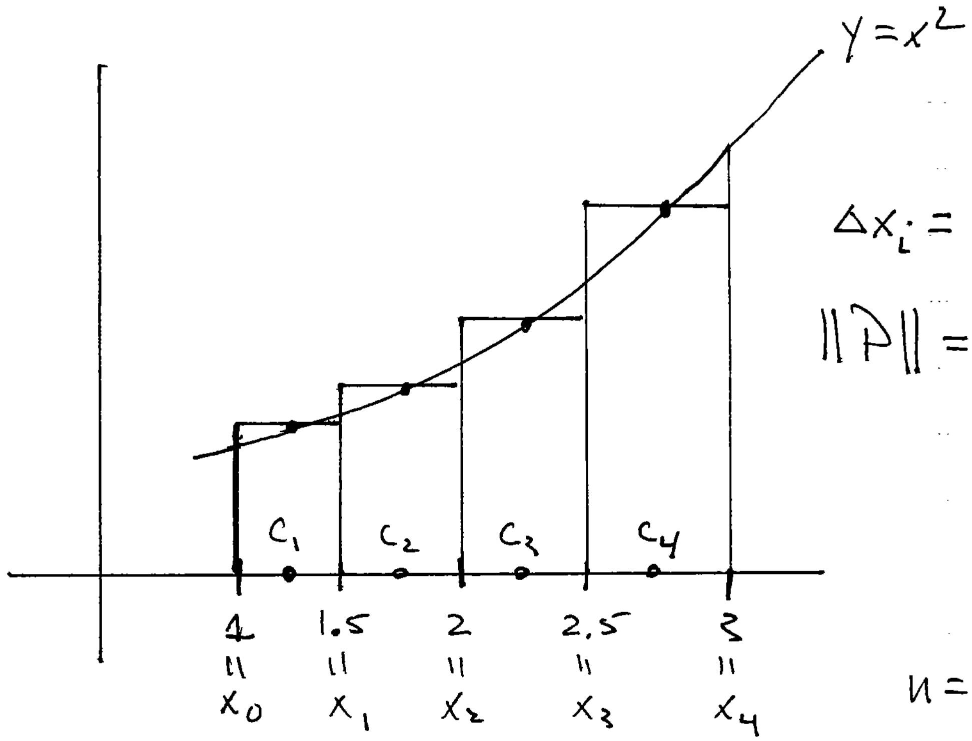
DETERMINE THE RIEMANN SUM FOR $f(x) = x^2$ ON $[1, 3]$ USING 4 SUBINTERVALS OF EQUAL WIDTHS, AND TAKING c_i TO BE MID POINTS OF THOSE INTERVALS.

THE PARTION P IS GIVEN BY

$$P = [1, 1.5, 2, 2.5, 3]$$

AND THE MIDPOINTS ARE:

1.25, 1.75, 2.25, 2.75



$\Delta x_i = .5$

$\|P\| = .5$

$n = 4$

$$S_P = \sum_{i=1}^{4} f(c_i)\Delta x_i$$

$$= (1.25)^2(.5) + (1.75)^2(.5) + (2.25)^2(.5) + (2.75)^2(.5)$$

$$= \dots = 8.625$$

TO OBTAIN BETTER APPROXIMATIONS WE CHOOSE FINER PARTITIONS, WHICH MEANS MORE SUBINTERVALS AND A SMALLER MESH $\|P\|$.

TO FIND AN EXACT VALUE FOR THE AREA, WE SIMULTANEOUSLY LET $n \to \infty$ AND $\|P\| \to 0$. WE INDICATE THIS SUCCINCTLY AS $\|P\| \to 0$.

THE LIMIT OF S_P AS $\|P\| \to 0$ (PROVIDED IT EXISTS) IS CALLED THE <u>DEFINITE INTEGRAL</u> OF f OVER $[a, b]$.

IT IS REQUIRED THAT THIS THE VALUE OF THIS LIMIT BE INDEPENDENT OF THE PARTICULAR CHOICE OF SEQUENCE OF PARTITIONS, AND OF THE CHOICE OF $c_i \in [x_{i-1}, x_i]$.

WE WRITE $\int_a^b f(x)\,dx$ FOR THIS

limit. i.e.

$$\boxed{\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i}$$

<u>NOTATION</u>: dx is A FORMAL SYMBOL WHICH MERELY SPECIFIES WHICH VARIABLE WE INTEGRATE WITH RESPECT TO.

$f(x)$ is CALLED THE <u>INTEGRAND</u>, AND a AND b ARE THE <u>UPPER</u> AND <u>LOWER</u> <u>limits OF INTEGRATION</u>, RESPECTIVELY.

<u>EX</u>. EXPRESS $\int_8^{10} \sqrt{x^2+1}\,dx$ AS A limit OF RIEMANN SUMS.

LET $x_0 = 8 < x_1 < x_2 < \ldots < x_n = 10$ DENOTE A SEQUENCE OF PARTIONS P WITH $n \to \infty$ AND $\|P\| \to 0$. THEN

$$\int_8^{10} \sqrt{x^2+1}\,dx = \lim_{\|P\| \to 0} \sum_{i=1}^{n} \sqrt{c_i^2+1}\,\Delta x_i$$

WHERE $c_i \in [x_{i-1}, x_i]$ ARE CHOSEN ARBITRARILY.

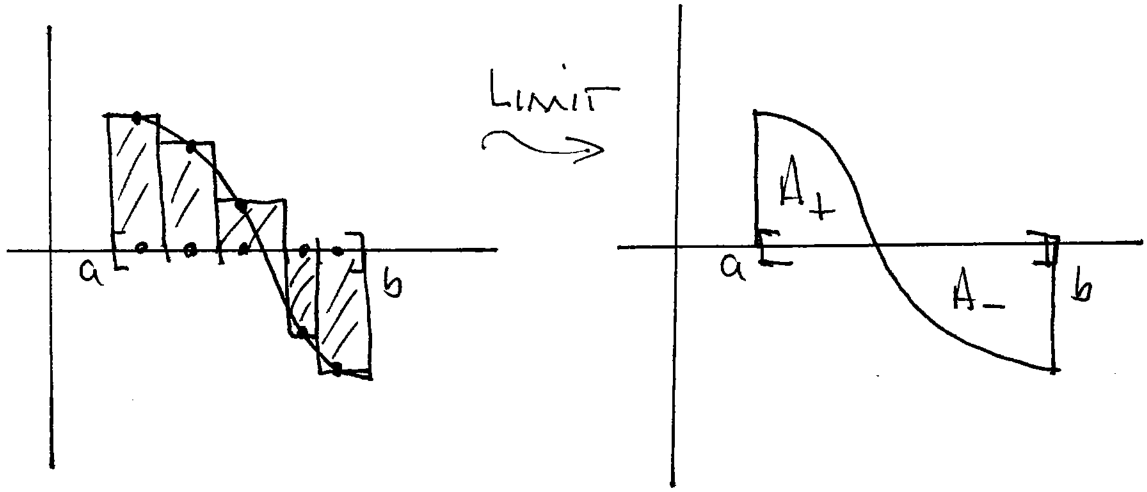
GEOMETRIC INTERPRETATION

AS LONG AS $f(x) \geq 0$ ON $[a,b]$ THE DEFINITE INTEGRAL

$$\int_a^b f(x)\,dx$$

CAN BE INTERPRETED AS THE AREA UNDER THE CURVE $y = f(x)$ FROM $x = a$ TO $x = b$.

IF $f(x) < 0$ FOR SOME $x \in [a,b]$ HOWEVER, THEN THE INTEGRAL SHOULD BE INTERPRETED AS A SIGNED AREA.



SOME RECTANGLES WILL HAVE A NEGATIVE HEIGHT, SO THOSE TERMS WILL SUBTRACT FROM THE TOTAL. THUS UPON TAKING THE LIMIT OF A SEQUENCE OF THESE RIEMANN SUMS, WE HAVE

$$\int_a^b f(x)\,dx = A_+ - A_-$$

WHERE A_+ IS THE AREA OF THE REGION ABOVE THE X-AXIS, AND A_- IS THE AREA OF THE REGION BELOW.

WE SEE THAT CERTAIN DEFINITE INTEGRALS CAN BE EVALUATED ON THE BASIS OF THIS GEOMETRIC INTERPRETATION ALONE

EX $$\int_0^3 (x-1)\,dx = 2 - \frac{1}{2} = \frac{3}{2}$$

EX $$\int_0^{2\pi} \cos x\,dx = 0$$

EX $$\int_0^3 \sqrt{9-x^2}\,dx = \frac{1}{4}\cdot(\pi\cdot 3^2) = \frac{9\pi}{4}$$