S.6. ANTIDERIVATIVES

We begin with a simple example.

Example: Determine a function $y = F(x)$ satisfying the condition

$$\frac{dy}{dx} = 3x^2$$

or equivalently

$$F'(x) = 3x^2$$

Observe that if $y = x^3$, then $\frac{dy}{dx} = 3x^2$ so $F(x) = x^3$ solves the problem.

Note also that $y = x^3 + 2$, and $y = x^3 + 50$ satisfy the given condition equally well. In fact, any function of the form

$$y = x^3 + c$$

where $c$ is an arbitrary constant, satisfies

$$\frac{dy}{dx} = 3x^2$$
Thus there are infinitely many valid answers to our original question.

**DEFN.**
A function $F(x)$ is said to be an antiderivative of $f(x)$ on an interval $I \subseteq \mathbb{R}$ if

$$F'(x) = f(x) \text{ for all } x \in I$$

We see that an antiderivative of $f(x)$ is any solution to a differential equation of the form

$$\frac{dy}{dx} = f(x)$$

And in general, such an equation has infinitely many solutions.

**EX.** Find antiderivatives of the following functions:

$$12x^5, 35x^6, 40x^7, \frac{y}{2}, x^2 + 2x - 1, e^x, e^{2x}, \cos x, \sin x, \sec^2 x$$

Note: Each of the above has infinitely many antiderivatives.
Recall Corollary 2 from Sec. 5.1

**Corollary (P.255)**

If \( f(x) \) is continuous on \([a, b]\), differentiable on \((a, b)\), and \( f'(x) = 0 \) for all \( x \in (a, b) \), then \( f(x) \) is constant on \([a, b]\).

I.e. there exist \( c \in \mathbb{R} \) such that \( f(x) = c \) for all \( x \in [a, b] \).

This corollary follows from the Mean Value Theorem. (P.255)

Notice this is the converse of the assertion that

\[
    f(x) = \text{const.} \quad \Rightarrow \quad f'(x) = 0
\]

This corollary has the following consequence.
**Corollary (P. 326)**

If \( F(x) \) and \( G(x) \) are two antiderivatives of \( f(x) \) on \( I \), then \( F(x) \) and \( G(x) \) differ by a constant.

i.e. There exists \( C \in \mathbb{R} \) such that

\[
G(x) = F(x) + C \quad \text{for all } x \in I.
\]

Thus to find all antiderivatives, one need find only one antiderivative.

**Proof**

Our hypotheses say that

\[
G'(x) = f(x) = F'(x),
\]

Whence

\[
0 = G'(x) - F'(x) = \frac{d}{dx} \left[ G(x) - F(x) \right],
\]

For all \( x \in I \). Applying the earlier corollary to the function \( G(x) - F(x) \), we obtain

\[
G(x) - F(x) = C
\]

For some constant \( C \).
Ex. Find the general antiderivative (i.e., all antiderivatives) of

\[ f(x) = x^4 - 3x^2 + 1 \]

We see that \( F(x) = \frac{1}{5} x^5 - x^3 + x \) is a particular antiderivative. Thus, all antiderivatives are of the form

\[ C_0(x) = \frac{1}{5} x^5 - x^3 + x + C \]

Here are some particular antiderivatives:

<table>
<thead>
<tr>
<th>Function</th>
<th>Antiderivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>( kf(x) )</td>
<td>( kF(x) )</td>
</tr>
<tr>
<td>( f(x) + g(x) )</td>
<td>( F(x) + G(x) )</td>
</tr>
<tr>
<td>( x^n ) (n ≠ -1)</td>
<td>( \frac{1}{n+1} x^{n+1} )</td>
</tr>
<tr>
<td>( \frac{1}{x} )</td>
<td>( \ln</td>
</tr>
<tr>
<td>( e^{ax} )</td>
<td>( \frac{1}{a} e^{ax} )</td>
</tr>
<tr>
<td>( \sin(ax) )</td>
<td>( -\frac{1}{a} \cos(ax) )</td>
</tr>
<tr>
<td>( \cos(ax) )</td>
<td>( \frac{1}{a} \sin(ax) )</td>
</tr>
<tr>
<td>( \sec^2(ax) )</td>
<td>( \frac{1}{a} \tan(ax) )</td>
</tr>
</tbody>
</table>
Ex: Graph the family of antiderivatives of \( f(x) = 2x \).

\[
\begin{align*}
  y &= f(x) = 2x \\
  y &= x^2 + 1 \\
  y &= x^2 \\
  y &= x^2 - 1 \\
  C(x) &= x^2 + C
\end{align*}
\]

All antiderivatives are obtained graphically by translating \( F(x) = x^2 \) vertically by an arbitrary distance \( C \).

Often, we wish to select a particular antiderivative which passes through a specific point \((x_0, y_0)\). This is called an initial value problem (IVP).

Such problems are usually stated as a differential equation.

Ex: \( \frac{dy}{dx} = 2x \) and \( y = -5 \) when \( x = 4 \)

Solve: \( y = x^2 - 21 \)
Ex. Find a function $N(t)$ satisfying

$$\frac{dN}{dt} = \frac{1}{2\sqrt{t}} \quad (t > 0)$$

and $N(0) = 20$

We have $N'(t) = \frac{1}{2} t^{-\frac{1}{2}}$ so that

$$N(t) = t^{\frac{1}{2}} + C = \sqrt{t} + C$$

Thus

$$20 = N(0) = \sqrt{0} + C$$

i.e. $C = 20$

i.e. $N(t) = \sqrt{t} + 20$

HW 5.8.1 (p. 330)

2-20 even, 38-48 even,
50-60 even, 66