CSE 201
Analysis of Algorithms
Winter 2021
Midterm Exam 1 **Solutions**

1. (20 Points) Suppose $f(n)$ and $g(n)$ are both asymptotically positive. Prove that if $f(n) = \Theta(g(n))$, then $1/f(n) = \Theta(1/g(n))$.

   Proof:
   We have positive c_1, c_2 and n_0 such that $0 < c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0$. Taking the reciprocal of all positive terms in this inquality gives

$$0 < \frac{1}{c_2 g(n)} \le \frac{1}{f(n)} \le \frac{1}{c_1 g(n)},$$

   and therefore

$$0 < \left(\frac{1}{c_2}\right) \cdot \frac{1}{g(n)} \le \frac{1}{f(n)} \le \left(\frac{1}{c_1}\right) \cdot \frac{1}{g(n)}$$

   for all $n \ge n_0$. It follows that $1/f(n) = \Theta(1/g(n))$. ■

2. (20 Points) Use Stirling's formula to prove that $\frac{(3n)!}{(n!)^3} = \Theta\left(\frac{27^n}{n}\right)$.

   Proof:

$$\frac{(3n)!}{(n!)^3} = \frac{\sqrt{2\pi \cdot 3n} \cdot \left(\frac{3n}{e}\right)^{3n} \cdot \left(1 + \Theta\left(\frac{1}{3n}\right)\right)}{\left(\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^{n} \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)\right)^3}$$

$$= \frac{\sqrt{3}}{2\pi} \cdot \frac{1}{n} \cdot \frac{3^{3n} \cdot n^{3n} \cdot e^{-3n}}{n^{3n} \cdot e^{-3n}} \cdot \frac{\left(1 + \Theta\left(\frac{1}{3n}\right)\right)}{\left(1 + \Theta\left(\frac{1}{n}\right)\right)^3}$$

$$= \frac{\sqrt{3}}{2\pi} \cdot \frac{27^n}{n} \cdot \frac{\left(1 + \Theta\left(\frac{1}{3n}\right)\right)}{\left(1 + \Theta\left(\frac{1}{n}\right)\right)^3}$$

   Therefore

$$\frac{\frac{(3n)!}{(n!)^3}}{\frac{27^n}{n}} = \frac{\sqrt{3}}{2\pi} \cdot \frac{\left(1 + \Theta\left(\frac{1}{3n}\right)\right)}{\left(1 + \Theta\left(\frac{1}{n}\right)\right)^3} \;\to\; \frac{\sqrt{3}}{2\pi} \quad \text{as } n \to \infty$$

   Since $0 < \sqrt{3}/2\pi < \infty$, it follows that $\frac{(3n)!}{(n!)^3} = \Theta\left(\frac{27^n}{n}\right)$. ■

3. (20 Points) The n^{th} harmonic number is defined to be he sum $H_n = \sum_{k=1}^{n} \left(\frac{1}{k}\right)$. Use induction to prove that for all $n \geq 1$:

$$\sum_{k=1}^{n} H_k = (n+1)H_n - n$$

(Hint: Use the fact that H_n satisfies the recurrence relation $H_n = H_{n-1} + \frac{1}{n}$.)

Proof:
Let $P(n)$ be the summation formula above.

I. Base step: $P(1)$.

If $n = 1$, then $H_1 = 1$ and $\sum_{k=1}^{1} H_k = 1 = 2 - 1 = (1+1) \cdot 1 - 1 = (1+1)H_1 - 1$, establishing the base case.

II. Induction form IIb: $\forall n > 1$: $P(n-1) \rightarrow P(n)$.

Let $n > 1$ be chosen arbitrarily, and assume $\sum_{k=1}^{n-1} H_k = ((n-1)+1)H_{n-1} - (n-1)$. We must show that $\sum_{k=1}^{n} H_k = (n+1)H_n - n$. We have

$$\sum_{k=1}^{n} H_k = \sum_{k=1}^{n-1} H_k \ + \ H_n$$

$$= \big((n-1)+1\big)H_{n-1} - (n-1) + H_n \qquad \text{by the induction hypothesis}$$

$$= nH_{n-1} - n + 1 + H_n$$

$$= nH_n - nH_n + nH_{n-1} - n + 1 + H_n$$

$$= (n+1)H_n - n + 1 - n(H_n - H_{n-1})$$

$$= (n+1)H_n - n + 1 - n\left(\frac{1}{n}\right) \qquad \text{by the recurrence for } H_n$$

$$= (n+1)H_n - n,$$

as required. If follows that $\sum_{k=1}^{n} H_k = (n+1)H_n - n$ for all $n \geq 1$. ■

4. (20 Points) Define $T(n)$ by the recurrence

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 4T(\lfloor n/2 \rfloor) + 2n^2 & \text{if } n \geq 2 \end{cases}$$

Use the substitution method to determine a constant $c > 0$ such that $T(n) \leq cn^2 \lg(n)$ for all $n \geq 1$ (and hence $T(n) = O(n^2 \log(n))$.

Solution:
Let $c = 2$. We show by induction that $\forall n \geq 1$: $T(n) \leq 2n^2 \lg(n)$, from which $T(n) = O(n^2 \log(n))$ follows.

I. For $n = 1$ we have $T(1) = 0 \leq 0 = 2 \cdot 1^2 \lg(1)$, establishing the base case

II. Let $n > 1$ be chosen arbitrarily, and assume $T(k) \leq 2k^2 \lg(k)$ for k in the range $1 \leq k < n$. We must show that $T(n) \leq 2n^2 \lg(n)$. Then

$$\begin{aligned} T(n) &= 4T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 2n^2 && \text{by the definition of } T(n) \\ &\leq 4 \cdot 2 \left\lfloor \frac{n}{2} \right\rfloor^2 \lg \left\lfloor \frac{n}{2} \right\rfloor + 2n^2 && \text{by the induction hypothesis with } k = \lfloor n/2 \rfloor \\ &\leq 8 \left(\frac{n}{2}\right)^2 \lg\left(\frac{n}{2}\right) + 2n^2 && \text{since } \lfloor x \rfloor \leq x \text{ for any } x \in \mathbb{R} \\ &= 2n^2(\lg(n) - 1) + 2n^2 \\ &= 2n^2 \lg(n) - 2n^2 + 2n^2 \\ &= 2n^2 \lg(n) \end{aligned}$$

The result follows for all $n \geq 1$ by the 2nd PMI. ∎

5. (20 Points) Let $T(n)$ satisfy the recurrence $T(n) = aT(n/b) + n^d$, where $a \geq 1$, $b > 1$, and $d > \log_b(a)$. Use the Master Theorem to find a tight asymptotic bound for $T(n)$. (Note your answer will depend on one or more of the constants a, b or d.)

Solution:
We compare n^d to $n^{\log_b a}$, and find that n^d is the winner since $d > \log_b(a)$, putting us in case 3 of the Master theorem. Let $\epsilon = d - \log_b(a)$, so $\epsilon > 0$ and $d = \log_b(a) + \epsilon$. Therefore

$$n^d = \Omega(n^d) = \Omega(n^{\log_b(a)+\epsilon}).$$

Observe also that $d > \log_b(a) \implies b^d > a \implies \frac{a}{b^d} < 1$. Pick any c in the range $\frac{a}{b^d} \leq c < 1$. Then for all $n \geq 1$ we have $a(n/b)^d = (a/b^d)n^d \leq cn^d$, establishing the regularity condition. Case 3 now gives us $T(n) = \Theta(n^d)$. ∎