CSE 201
Homework Assignment 1
Solutions

1. (Problem 3.1-1) Let $f(n)$ and $g(n)$ asymptotically positive functions. Prove that $f(n) + g(n) = \Theta(\max(f(n),\ g(n)))$.

   Proof:
   Since $f(n)$ and $g(n)$ are asymptotically positive, we know that there exists a positive constant n_0 such that $f(n) > 0$ and $g(n) > 0$ for all $n \geq n_0$. For any such n we have

$$\begin{aligned} 0 &\leq \max\bigl(f(n), g(n)\bigr) \\ &\leq \min\bigl(f(n), g(n)\bigr) + \max\bigl(f(n), g(n)\bigr) \\ &\leq 2 \cdot \max\bigl(f(n), g(n)\bigr). \end{aligned}$$

   But $f(n) + g(n) = \min\bigl(f(n), g(n)\bigr) + \max\bigl(f(n), g(n)\bigr)$, and therefore

$$0 \leq 1 \cdot \max\bigl(f(n), g(n)\bigr) \leq f(n) + g(n) \leq 2 \cdot \max\bigl(f(n), g(n)\bigr)$$

   for all $n \geq n_0$, showing that $f(n) + g(n) = \Theta(\max\,(f(n), g(n)))$. ■

2. Prove or disprove: If $f(n) = \Theta(g(n))$, then $f(n)^2 = \Theta(g(n)^2)$.

   Proof:
   Since $f(n) = \Theta(g(n))$, there are positive constants c_1, c_2 and n_0 such that

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$$

   for all $n \geq n_0$. Squaring this inequality gives $0 \leq c_1^2 g(n)^2 \leq f(n)^2 \leq c_2^2 g(n)^2$, showing that $f(n)^2 = \Theta(g(n)^2)$. ■

3. Prove or disprove: If $f(n) = \Theta(g(n))$, then $2^{f(n)} = \Theta(2^{g(n)})$.

   Counter Example:
   Let $f(n) = 2n$ and $g(n) = n$. Then $f(n) = \Theta(g(n))$, but $2^{2n} = 4^n = \omega(2^n)$, and therefore $2^{f(n)} = \omega(2^{g(n)})$, whence $2^{f(n)} \neq \Theta(2^{g(n)})$. ■

4. Let $f(n)$ and $g(n)$ be asymptotically positive functions, and assume that $\lim_{n\to\infty} g(n) = \infty$. Prove that if $f(n) = \Theta(g(n))$, then $\ln\bigl(f(n)\bigr) = \Theta\bigl(\ln\bigl(g(n)\bigr)\bigr)$.

   Proof:
   Assume $f(n) = \Theta(g(n))$. Then there exist positive constants c_1, c_2 and n_0 such that

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$$

for all $n \geq n_0$. Since $\lim_{n\to\infty} g(n) = \infty$, the constant n_0 can be chosen large enough so that

$$1 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$$

for all $n \geq n_0$. Take ln() of all terms in the preceding inequality to get

$$0 \leq \ln(c_1) + \ln\big(g(n)\big) \leq \ln\big(f(n)\big) \leq \ln(c_2) + \ln\big(g(n)\big)$$

for all $n \geq n_0$. Since $\lim_{n\to\infty} g(n) = \infty$, the term $\ln\big(g(n)\big)$ dominates the constants $\ln(c_1)$ and $\ln(c_2)$, whence $\ln(c_1) + \ln\big(g(n)\big) = \Omega\big(\ln\big(g(n)\big)\big)$ and $\ln(c_2) + \ln\big(g(n)\big) = O\big(\ln\big(g(n)\big)\big)$. The desired result $\ln\big(f(n)\big) = \Theta\big(\ln\big(g(n)\big)\big)$ now follows by Exercise 4 on page 4 of the handout on asymptotic growth rates. ■

5. (Problem 3.2-8) Show that if $f(n)\ln f(n) = \Theta(n)$, then $f(n) = \Theta(n/\ln n)$. Hint: use the result of the preceding problem.

Proof:
Assume $f(n)\ln f(n) = \Theta(n)$. Since $\lim_{n\to\infty} n = \infty$, we can, by the result of problem (4), take ln () of both sides of this equation to get

$$\ln\big(f(n)\big) + \ln\big(\ln\big(f(n)\big)\big) = \Theta(\ln(n)).$$

Since the term $\ln\big(\ln\big(f(n)\big)\big) = o(\ln\big(f(n)\big)$, we can, by Exercise 9g on page 8 of the handout on asymptotic growth rates, drop the lower order term and write this as $\ln\big(f(n)\big) = \Theta(\ln(n))$. Substituting this into our assumption gives $f(n)\Theta(\ln n) = \Theta(n)$, and therefore

$$f(n) = \Theta(n) \cdot \frac{1}{\Theta(\ln n)} = \Theta(n) \cdot \Theta\left(\frac{1}{\ln n}\right) = \Theta\left(\frac{n}{\ln n}\right)$$

where we have used Exercise 11bc on page 8 of the handout on asymptotic growth rates. ■

6. List the following 27 functions from lowest to highest asymptotic order. Indicate whether any two (or more) are of the same asymptotic order. Justify your answers.

$$2^{\lg(n)}, (\lg n)^{\lg n}, n^{1/\lg n}, \ln(\ln(n)), 4^{\lg n}, \sqrt{2}^{\lg n}, n, \sqrt{2n}, 1$$
$$n^2, n!, (3/2)^n, n^3, (\lg n)^2, 2^{2^n}, n2^n, n^{\lg(\lg n)}, \ln(n), e^n$$
$$2^{\sqrt{2\lg n}}, 2^n, n\lg n, 2^{2^{n+1}}, \lg(n!), \sqrt{\lg n}, 2^{(\lg n)^2}, (n+1)!$$

Solution:
The functions listed in increasing order are:

(1) $1, n^{1/\lg n}$
(2) $\ln(\ln(n))$
(3) $\sqrt{\lg n}$
(4) $\ln(n)$

(5) $(\lg n)^2$
(6) $2^{\sqrt{2\lg n}}$
(7) $\sqrt{2n}$, $\sqrt{2}^{\lg n}$
(8) n, $2^{\lg(n)}$
(9) $n\lg n$, $\lg(n!)$
(10) n^2, $4^{\lg n}$
(11) n^3
(12) $n^{\lg(\lg n)}$, $(\lg n)^{\lg n}$
(13) $2^{(\lg n)^2}$
(14) $(3/2)^n$
(15) 2^n
(16) $n2^n$
(17) e^n
(18) $n!$
(19) $(n+1)!$
(20) 2^{2^n}
(21) $2^{2^{n+1}}$

Justifications:

(1) Let $y = n^{1/\lg n}$, then $\lg y = \lg\left(n^{1/\lg n}\right) = (1/\lg n)\cdot \lg n = 1$, hence $y = 2$ and $n^{1/\lg n} = 2 = \Theta(1)$.

(2) $\lim\limits_{n\to\infty}(\ln(\ln n)/1) = \infty$, so that $\ln(\ln n) = \omega(1)$.

(3) Let $y_n = \sqrt{\lg n}/\ln(\ln n)$. Then $\lg y_n = \frac{1}{2}\lg(\lg n) - \lg(\ln(\ln n)) \to \infty$ as $n\to\infty$, hence $y_n \to \infty$ as $n \to \infty$, and therefore $\sqrt{\lg n} = \omega(\ln(\ln n))$.

(4) Observe $\frac{\ln n}{\sqrt{\lg n}} = \frac{c\lg n}{\sqrt{\lg n}} = c\cdot\sqrt{\lg n} \to \infty$, where c is a constant, so $\ln n = \omega(\sqrt{\lg n})$.

(5) We have $(\lg n)^2/\ln n = (\lg n)^2/c\lg n = \lg n/c \to \infty$, and hence $(\lg n)^2 = \omega(\ln n)$.

(6) Let $y_n = 2^{\sqrt{2\lg n}}/(\lg n)^2$. Then $\lg y_n = \sqrt{2}\cdot\sqrt{\lg n} - 2\lg(\lg n) \to \infty$, showing $y_n \to \infty$, and therefore $2^{\sqrt{2\lg n}} = \omega((\lg n)^2)$.

(7) Observe that $\sqrt{2}^{\lg n} = n^{\lg\sqrt{2}} = n^{1/2} = \sqrt{n} = \Theta(\sqrt{2n})$. Now let $y_n = \sqrt{n}/2^{\sqrt{2\lg n}}$. Then $\lg y_n = \frac{1}{2}\lg n - \sqrt{2}\cdot\sqrt{\lg n} \to \infty$, so $y_n \to \infty$, and $\sqrt{n} = \omega(2^{\sqrt{2\lg n}})$.

(8) Note $2^{\lg n} = n^{\lg 2} = n = \omega(\sqrt{n})$.

(9) Recall we showed $\lg(n!) = \Theta(n\lg n)$ using Stirling's formula. Also $n\lg n/n = \lg n \to \infty$, so that $n\lg n = \omega(n)$.

(10) We have $4^{\lg n} = n^{\lg 4} = n^2$, and $n^2/n\lg n = n/\lg n \to \infty$ (by l'Hopital's rule), hence $n^2 = \omega(n\lg n)$.

(11) Clearly $n^3 = \omega(n^2)$, since $n^3/n^2 = n \to \infty$.

(12) Observe $(\lg n)^{\lg n} = n^{\lg(\lg n)}$. Note $n^{\lg(\lg n)}/n^3 = n^{\lg(\lg n)-3} \to \infty$, since $(\lg(\lg n) - 3) \to \infty$, showing that $n^{\lg(\lg n)} = \omega(n^3)$.

(13) Let $y_n = 2^{(\lg n)^2}/n^{\lg(\lg n)}$. Then $\lg y_n = (\lg n)^2 - (\lg\lg n)\cdot(\lg n) \to \infty$, whence $y_n \to \infty$, and therefore $2^{(\lg n)^2} = \omega(n^{\lg(\lg n)})$.

(14) Let $y_n = (3/2)^n/2^{(\lg n)^2}$. Then $\lg y_n = n\lg(3/2) - (\lg n)^2 \to \infty$, so that $y_n \to \infty$, and hence $(3/2)^n = \omega(2^{(\lg n)^2})$.

(15) We have $2^n = \omega((3/2)^n)$, since $2 > 3/2$.

(16) $n2^n = \omega(2^n)$ since $n2^n/2^n = n \to \infty$.

(17) Observe $e^n = \omega(n2^n)$, since $\lim_{n\to\infty} \frac{e^n}{n2^n} = \lim_{n\to\infty} \frac{(e/2)^n}{n} = \lim_{n\to\infty} \frac{\text{const}\cdot(e/2)^n}{1} = \infty$, using l`Hopital's rule and the fact that $e > 2$.

(18) Let $y_n = n!/e^n$. Then $\lg y_n = \lg(n!) - n\, lg\, e = \Theta(n\, lg\, n) - \Theta(n) \to \infty$, whence $y_n \to \infty$, and therefore $n! = \omega(e^n)$.

(19) We have $(n+1)!/n! = (n+1) \to \infty$, so that $(n+1)! = \omega(n!)$.

(20) Let $y_n = 2^{2^n}/(n+1)!$. Then $\lg y_n = 2^n - \lg((n+1)!) = 2^n - \Theta((n+1)\lg(n+1) = 2^n - \Theta(n\lg n)$, therefore $\lg y_n \to \infty$, hence $y_n \to \infty$, showing that $2^{2^n} = \omega((n+1)!)$.

(21) Observe $2^{n+1} - 2^n = 2^n(2-1) = 2^n \to \infty$, so that $2^{2^{n+1}}/2^{2^n} = 2^{2^{n+1}-2^n} \to \infty$, showing that $2^{2^{n+1}} = \omega(2^{2^n})$. ■

7. Determine the asymptotic order of the expression $\sum_{i=1}^{n} a^i$ where $a > 0$ is a constant, i.e. find a simple function $g(n)$ such that the expression is in the class $\Theta(g(n))$. (Hint: consider the cases $a = 1$, $a > 1$, and $0 < a < 1$ separately.)

Solution: In the case $a = 1$ we have $\sum_{i=1}^{n} a^i = n = \Theta(n)$. Now assume $a \neq 1$ and use the formula for the sum of a geometric series:

$$\sum_{i=1}^{n} a^i = a\left(\sum_{i=0}^{n-1} a^i\right) = a\left(\frac{a^n - 1}{a - 1}\right) = \left(\frac{a}{a-1}\right)\cdot a^n - \left(\frac{a}{a-1}\right)$$

If $a > 1$ then $a^n \to \infty$ as $n \to \infty$, so the first term dominates the second, and $\sum_{i=1}^{n} a^i = \Theta(a^n)$. If $0 < a < 1$ then $a^n \to 0$, so the constant term (which is positive in this case) dominates, and therefore $\sum_{i=1}^{n} a^i = \Theta(1)$. ■

8. Use induction to prove that $\sum_{k=1}^{n} k^4 = \frac{n(n+1)(6n^3+9n^2+n-1)}{30}$ for all $n \geq 1$.

Proof:

I. $\sum_{k=1}^{1} k^4 = 1 = \frac{2\cdot 15}{30} = \frac{1\cdot(1+1)\cdot(6\cdot 1^3+9\cdot 1^2+1-1)}{30}$, and the base case is established.

II. Let $n \geq 1$. Assume for this *n* that

$$\sum_{k=1}^{n} k^4 = \frac{n(n+1)(6n^3+9n^2+n-1)}{30}$$

We must show that

$$\sum_{k=1}^{n+1} k^4 = \frac{(n+1)(n+2)(6(n+1)^3+9(n+1)^2+(n+1)-1)}{30}$$

We have

$$\sum_{k=1}^{n+1} k^4 = \left(\sum_{k=1}^{n} k^4\right) + (n+1)^4$$

$$= \frac{n(n+1)(6n^3+9n^2+n-1)}{30} + (n+1)^4 \qquad \text{(by the induction hypothesis)}$$

$$= \frac{(n+1)\left[n(6n^3+9n^2+n-1)+30(n+1)^3\right]}{30}$$

$$= \frac{(n+1)\left[6n^4+39n^3+91n^2+89n+30)\right]}{30}$$

$$= \frac{(n+1)(n+2)\left[6n^3+27n^2+37n+15)\right]}{30}$$

$$= \frac{(n+1)((n+1)+1)\left[6(n+1)^3+9(n+1)^2+(n+1)-1)\right]}{30}$$

Thus the formula holds for $n+1$ as well. The formula is valid for all n by induction. ■