CSE 201 2-23-21

- Exam 2 on thursday

continue .. Matrix Chain mult. ...

Greedy Algorithms

Ex. Continuous Knapsack

Goal:

(1) maximize: $\sum_{i=1}^{n} x_i v_i$ ← objective function

(2) subject to: $\sum_{i=1}^{n} x_i w_i \leq W$

where $W > 0$, $v_i > 0$, $w_i > 0$ $\{1 \leq i \leq n\}$.

Also $0 \leq x_i \leq 1$ $(1 \leq i \leq n)$. (instead of $x_i \in \{0, 1\}$ for discrete version.)

A vector $x = (x_1, x_2, \ldots, x_n)$ is called a feasible solution if (2) is satisfied without regard to (1)

Defn

A greedy strategy consists of making some (locally) optimal choice, then solving the sub-instance resulting from that choice.

for cont. knapsack: grab the "best" object first.

what does "best" mean?

see Java code for greedy knapsack.

In general, we must define a <u>selection function</u> $f(i)$ that ranks objects $i = 1, \ldots, n$ by their desirability.

Possible selection fcns for C.K.P.

- $f(i) = v_i$: sort objects by decreasing values.
- $f(i) = \frac{1}{w_i}$: sort objects by increasing weight.
- $f(i) = \frac{v_i}{w_i}$: sort objects by decreasing value density.

<u>Ex.</u> $n = 5$, $v = (2, 3, 6.6, 4, 6)$

$w = (1, 2, 3, 4, 5)$, $W = 10$

i	1	2	3	4	5
v_i	2	3	6.6	4	6
w_i	1	2	3	4	5
$\frac{v_i}{w_i}$	2	1.5	2.2	1	1.2

$f(i)$	x					Value
v_i	0	0	1	.5	1	14.6
$\frac{1}{w_i}$	1	1	1	1	0	15.6
$\frac{v_i}{w_i}$	1	1	1	0	.8	16.4

Theorem

If we maximize $\frac{v_i}{w_i}$ in our greedy algorithm (i.e. sort by decreasing value densities.), then the greedy algorithm returns an optimal solution.

Proof.

Assume objects are already sorted by desireability:

$$\frac{v_1}{w_1} \geq \frac{v_2}{w_2} \geq \frac{v_3}{w_3} \geq \cdots\cdots \geq \frac{v_n}{w_n}$$

Let $x = (x_1, x_2, x_3, \ldots, x_n)$ be the soln. returned by the greedy algorithm.

If all $x_i = 1$, then obviously x is optimal. otherwise, let j be first index for which $x_j < 1$.

Thus we have (from the algorithm)

$$x_i = 1 \quad \text{for} \quad 1 \le i < j$$

$$x_j < 1$$

$$x_i = 0 \quad \text{for} \quad j < i \le n$$

Also

$$\sum_{i=1}^{n} x_i w_i = W .$$

Let $y = (y_1, y_2, y_3, \dots, y_n)$ be any feasible solution, i.e.

$$\sum_{i=1}^{n} y_i w_i \le W ,$$

<u>notation</u>:

$$V(y) = \sum_{i=1}^{n} y_i v_i \qquad (\text{value of } y)$$

$$V(x) = \sum_{i=1}^{n} x_i v_i \qquad (\text{value of } x).$$

we must show that

$$V(x) \geq V(y).$$

we have

$$\sum_{i=1}^{n} (x_i - y_i) w_i = \sum_{i=1}^{n} x_i w_i - \sum_{i=1}^{n} y_i \cdot w_i$$

$$= W - \sum_{i=1}^{n} y_i w_i \geq 0$$

Also

$$V(x) - V(y) = \sum_{i=1}^{n} (x_i - y_i) v_i$$

$$= \sum_{i=1}^{n} (x_i - y_i) \cdot w_i \cdot \left(\frac{v_i}{w_i}\right)$$

observe

- $i < j \Rightarrow x_i = 1 \Rightarrow (x_i - y_i) \geq 0$ & $\frac{v_i}{w_i} \geq \frac{v_j}{w_j}$

  $\therefore (x_i - y_i)\left(\frac{v_i}{w_i}\right) \geq (x_i - y_i)\left(\frac{v_j}{w_j}\right)$

- $i > j \Rightarrow x_i = 0 \Rightarrow (x_i - y_i) \leq 0$ & $\frac{v_i}{w_i} \leq \frac{v_j}{w_j}$

  $\therefore (x_i - y_i)\left(\frac{v_i}{w_i}\right) \geq (x_i - y_i)\left(\frac{v_j}{w_j}\right)$

- $i = j \Rightarrow (x_i - y_i)\left(\frac{v_i}{w_i}\right) = (x_i - y_i)\left(\frac{v_j}{w_j}\right)$

In all cases:

$$(x_i - y_i)\left(\frac{v_i}{w_i}\right) \geq (x_i - y_i)\left(\frac{v_j}{w_j}\right) \qquad (1 \leq i \leq n)$$

$$\therefore V(x) - V(y) = \sum_{i=1}^{n} (x_i - y_i) \cdot \left(\frac{v_i}{w_i}\right) \cdot w_i$$

$$\geq \sum_{i=1}^{n} (x_i - y_i)\left(\frac{v_j}{w_j}\right) w_i$$

$$= \left(\frac{v_j}{w_j}\right) \cdot \sum_{i=1}^{n} (x_i - y_i) w_i$$

$$\geq 0$$

$\therefore \quad V(x) \geq V(y)$

$\therefore \quad x$ is optimal !!! $\blacksquare$