CSE 201 1-7-21

continue ...

Exercise

Prove that $f(n) = \Theta(g(n))$ iff $g(n) = \Theta(f(n))$.

Exercise

let $g(n)$ be a fcn., let $c > 0$. show

- $c \cdot g(n) = O(g(n))$
- $c \cdot g(n) = \Omega(g(n))$
- $c \cdot g(n) = \Theta(g(n))$

Ex. Prove $\sqrt{n+10} = \Theta(\sqrt{n})$

Proof.

The defn. says we must find positive c_1, c_2, n_0 s.t. $\forall n \geq n_0$:

$$0 \leq c_1\sqrt{n} \leq \sqrt{n+10} \leq c_2\sqrt{n} \quad .$$

let $c_1 = 1$, $c_2 = \sqrt{2}$, $n_0 = 10$. Then if $n \geq n_0$, we have

$$-10 \leq 0 \quad \text{and} \quad 10 \leq n$$

$$\therefore \quad -10 \leq (1-1)n \quad \text{and} \quad 10 \leq (2-1)n$$

$$\therefore \quad -10 \leq (1-c_1^2)n \quad \text{and} \quad 10 \leq (c_2^2-1)n$$

$$\therefore \quad c_1^2 n \leq n+10 \quad \text{and} \quad n+10 \leq c_2^2 n$$

$$\therefore \quad c_1^2 n \leq n+10 \leq c_2^2 n$$

$$\therefore \quad 0 \leq c_1\sqrt{n} \leq \sqrt{n+10} \leq c_2\sqrt{n} \quad ,$$

which completes the proof. ∎

Exercise let $a, b \in \mathbb{R}$ with $b > 0$.
Prove that

$$(n+a)^b = \Theta(n^b)$$

Theorem

If $h(n) = O(g(n))$ and if $f(n) \leq h(n)$ for all suff. large n, then $f(n) = O(g(n))$.

Proof:
our hypotheses say: there exist pos. numbers c_1, n_1 s.t. for all $n \geq n_1$

$$(1) \qquad 0 \leq h(n) \leq c_1 g(n)$$

also there exists pos. n_2 s.t. $\forall n \geq n_2$,

$$(2) \qquad f(n) \leq h(n).$$

we must find pos. c, n_0 s.t. $\forall n \geq n_0$

$$(3) \qquad 0 \leq f(n) \leq c \cdot g(n)$$

Let $c = c_1$ and $n_0 = \max(n_1, n_2)$.

If $n \geq n_0$, then both $n \geq n_1$ and $n \geq n_2$, so both (1) and (2) are true, hence (3) is true. $\blacksquare$

<u>Exercise:</u>
If $h(n) = \Omega(g(n))$ and if $f(n) \geq h(n)$ for all suff. large n, then $f(n) = \Omega(g(n))$.

<u>Exercise</u>
if $h_1(n) \leq f(n) \leq h_2(n)$ for all suff. large n, where $h_1(n) = \Omega(g(n))$ and $h_2(n) = O(g(n))$, then $f(n) = \Theta(g(n))$.

Ex. let $k \geq 0$ be fixed real number.

show $\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$

remark: we have formulas

$k=1$: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$k=2$: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

$k=3$: $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

$\vdots$

Proof:

observe that

$$\sum_{i=1}^{n} i^K \leq \sum_{i=1}^{n} n^K = n \cdot n^K = n^{K+1} = O(n^{K+1}).$$

also

$$\sum_{i=1}^{n} i^K \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^K$$

$$\geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \lceil \tfrac{n}{2} \rceil^K$$

$$= \left(n - \lceil \tfrac{n}{2} \rceil + 1\right) \lceil \tfrac{n}{2} \rceil^K$$

$$= \left(\lfloor \tfrac{n}{2} \rfloor + 1\right) \lceil \tfrac{n}{2} \rceil^K$$

$$> \left(\tfrac{n}{2} - \cancel{1} + \cancel{1}\right) \left(\tfrac{n}{2}\right)^K$$

$$\left\{ \begin{array}{l} \text{since:} \\ \lfloor x \rfloor > x - 1 \\ \lceil x \rceil \geq x \end{array} \right.$$

$$= \left(\tfrac{n}{2}\right)^{K+1}$$

$$= \left(\tfrac{1}{2}\right)^{K+1} \cdot n^{K+1} = \Omega(n^{K+1}).$$

$$\therefore \sum_{i=1}^{n} i^K = \Theta(n^{K+1}).$$

■

Exercise: $\underbrace{\sum_{i=1}^{n} \Theta(i) = \Theta(n^2)}$

what does this mean?

Doesn't mean:

$$\Theta(1) + \Theta(2) + \dots + \Theta(n)$$

it means

$$f(1) + f(2) + \dots + f(n)$$

for some fcn. $f(i) = \Theta(i)$.

so show

$$\sum_{i=1}^{n} f(i) = \Theta(n^2).$$

Defn

$$o(g(n)) = \{f(n) \mid \forall c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq f(n) < c\,g(n)\}$$

we say $g(n)$ is a strict asymptotic u.b. for $f(n)$.

Recall

$$O(g(n)) = \{f(n) \mid \exists c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq f(n) \leq c\,g(n)\}$$

observe that:

$$o(g(n)) \subseteq O(g(n)).$$

Lemma : $f(n) = o(g(n))$ iff $\lim_{n\to\infty} \left(\frac{f(n)}{g(n)}\right) = 0$

Proof: observe $f(n) = o(g(n))$ iff

$$\forall c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq \frac{f(n)}{g(n)} < c.$$

This is the very defn. of $\lim_{n\to\infty} \left(\frac{f(n)}{g(n)}\right) = 0$.

∎

Ex. $\ln(n) = o(n)$ ✓ why?

$$\lim_{n\to\infty}\left(\frac{\ln(n)}{n}\right) = \lim_{n\to\infty}\left(\frac{1/n}{1}\right) = 0$$

Ex. $(\ln(n))^a = o(n^b)$ where $a, b \geq 0$ are integers. use l'hop. rule multiple times.

Observe: all logs $<$ all Poly

grow slower than

Ex $n^k = o(e^n)$ ✓

$$\lim_{n\to\infty}\left(\frac{n^k}{e^n}\right) = k\lim_{n\to\infty}\left(\frac{n^{k-1}}{e^n}\right)$$

$$= \cdots\cdots = k(k-1)\cdots(2)\cdot 1\cdot\lim_{n\to\infty}\left(\frac{1}{e^n}\right) = 0$$

Analogy:

$f(n) = O(g(n)) \sim x \leq y$

$f(n) = \Omega(g(n)) \sim x \geq y$

$\Theta \sim =$

$o \sim <$

$\omega \sim >$