CSE 201 1-5-21

Asymptotic growth rates of functions

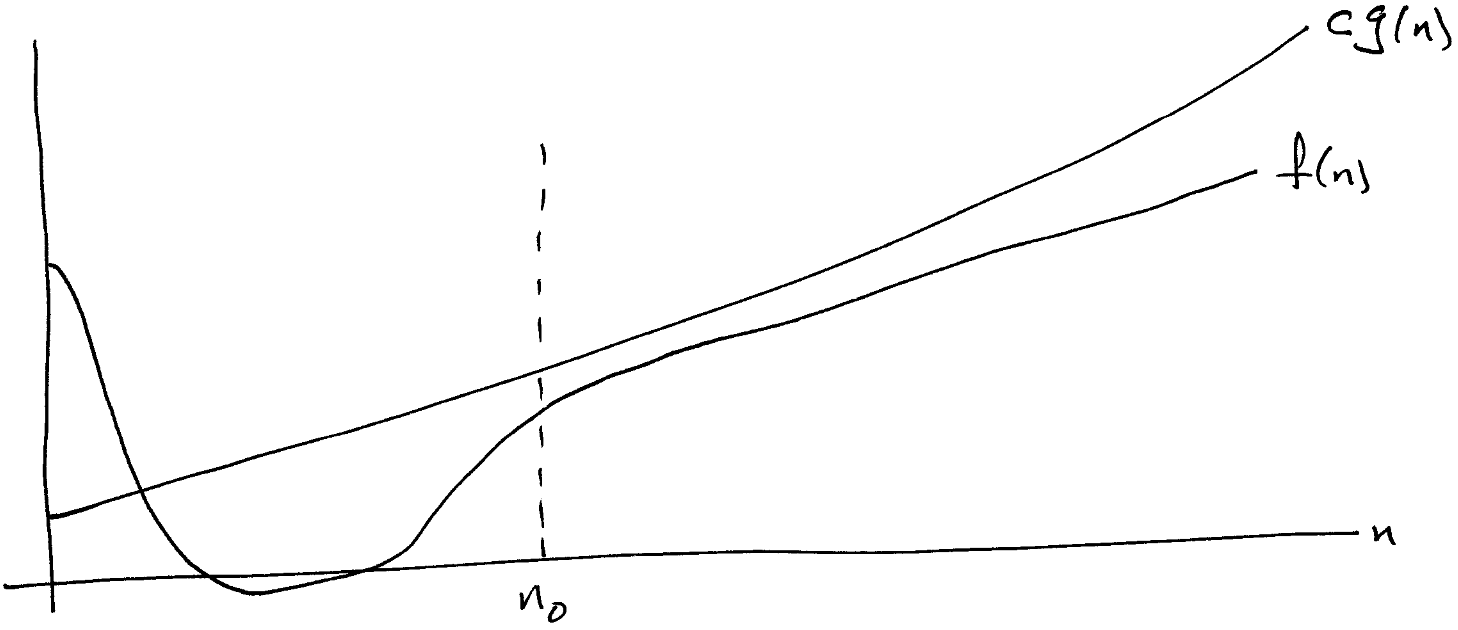
Defn

let $g(n)$ be a function.

$$O(g(n)) = \{ f(n) \mid \exists c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq f(n) \leq c g(n) \}$$

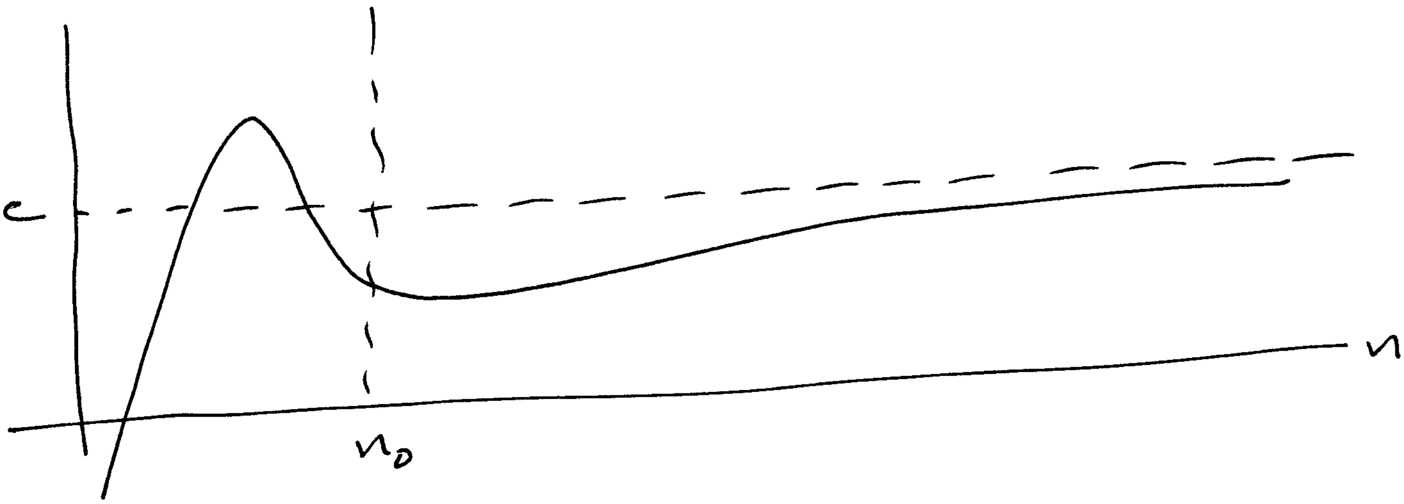
we say: $f(n)$ is Big O of $g(n)$

" : $g(n)$ is an asymptotic upper bound for $f(n)$



equivalently:

$$\exists c > 0,\ \exists n_0 > 0\ \forall n \ge n_0 : \quad 0 \le \frac{f(n)}{g(n)} \le c$$



<u>Defn</u> $f(n)$ is asymptotically non-negative (a.n.n

iff

$$\exists n_0 > 0,\ \forall n \ge n_0 : f(n) \ge 0 .$$

<u>Defn</u> $f(n)$ is asymptotically positive (a.p)

iff

$$\exists n_0 > 0\ \forall n \ge n_0 : f(n) > 0 .$$

<u>Blanket assumption</u>: all fcns. under discussion are a.n.n.

(occasionally we will need a.p.)

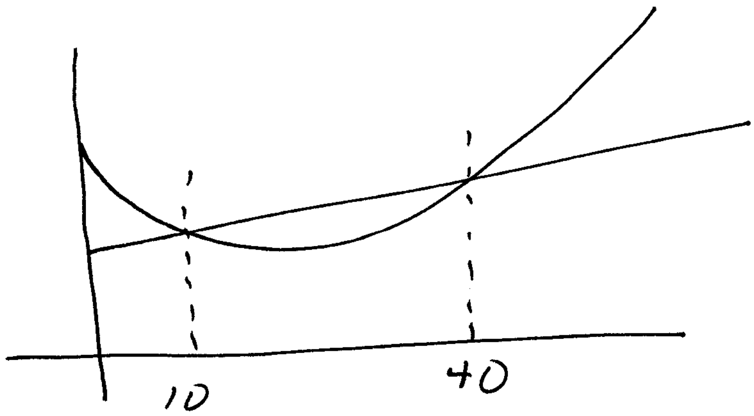
<u>Ex.</u> $10n + 100 = O(n^2 - 40n + 500)$

$\uparrow$ means $\in$

why? observe

$$0 \le (10n + 100) \le (n^2 - 40n + 50)$$

for all $n \ge 40$.



In this case $n_0 = 40$, $c = 1$.

In fact $an+b = O(cn^2+dn+e)$ for any a, b, c, d, e with $a>0$, $c>0$.

Also $p(n) = O(q(n))$ whenever $p(n), q(n)$ are polynomials with

$$\deg(p(n)) \leq \deg(q(n)).$$

<u>Defn</u>

$\Omega(g(n)) = \{f(n) \mid \exists c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq c\,g(n) \leq f(n)\}$

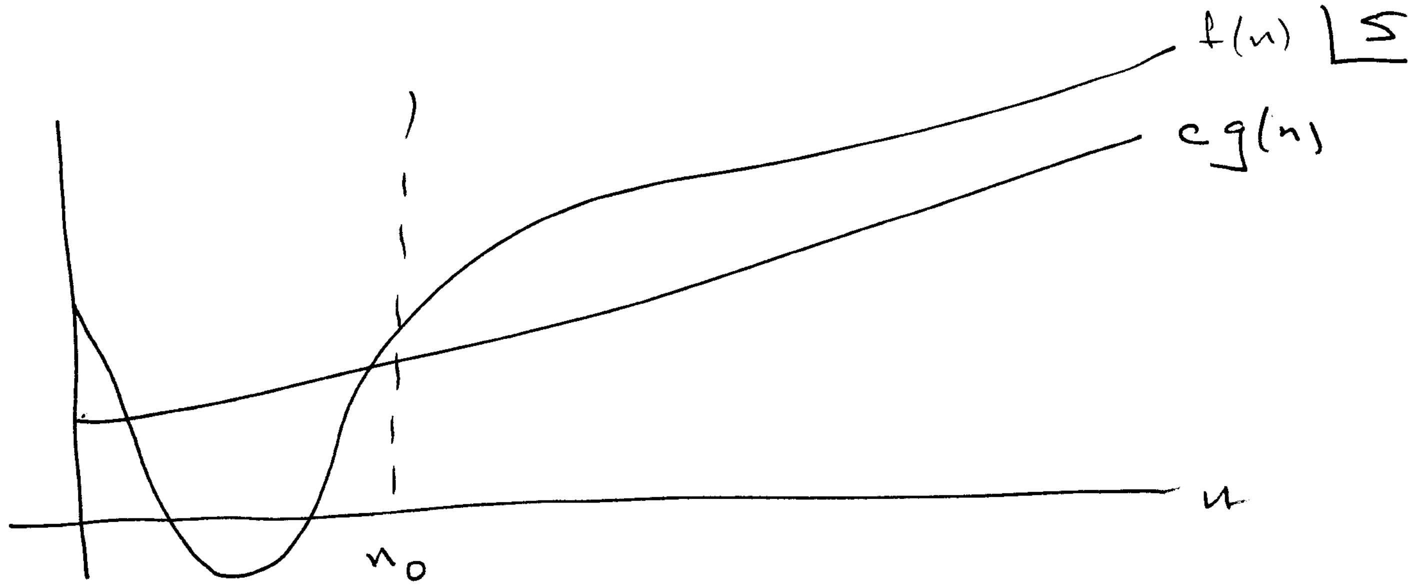
we say $f(n)$ is big omega of $g(n)$

" " $g(n)$ is an asym. lower bound.

We write : $f(n) = \Omega(g(n))$

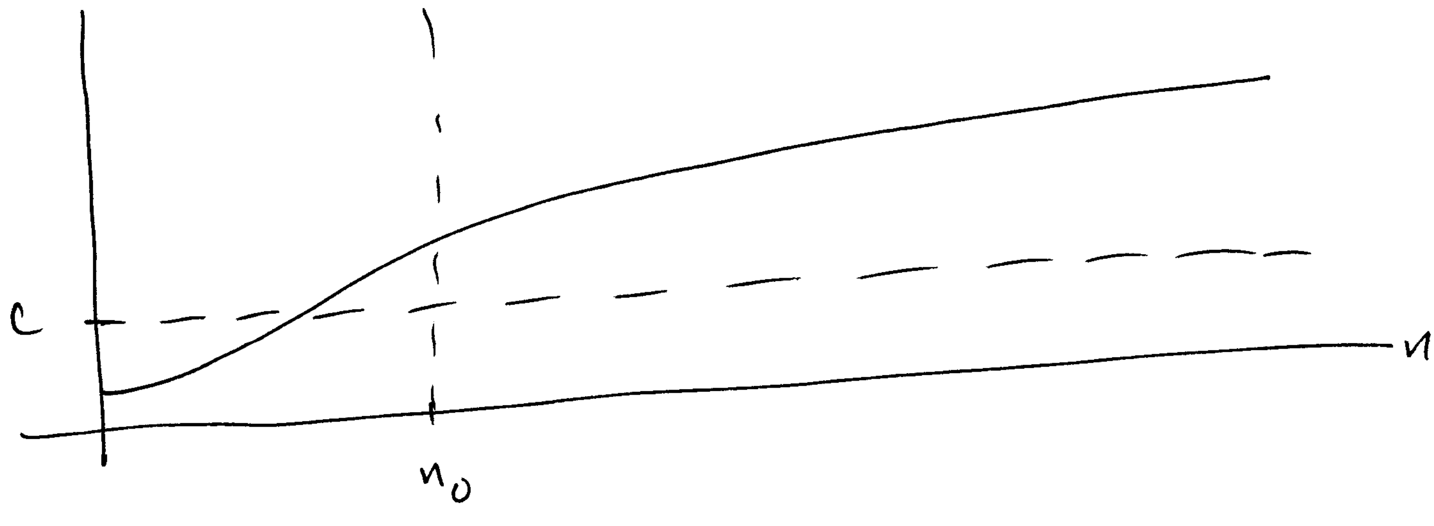
$\uparrow$

means $f(n) \in \Omega(g(n))$.



equivalently:

$$\forall n \geq n_0 \quad : \quad 0 \leq c \leq \frac{f(n)}{g(n)}$$



Theorem

$$f(n) = O(g(n)) \quad \underset{\Longleftarrow}{\overset{\Longrightarrow}{\text{iff}}} \quad g(n) = \Omega(f(n)).$$

Proof of ($\Rightarrow$):

Suppose $f(n) = O(g(n))$. Then there exist positive c_1, n_1 s.t.

$$(1) \qquad \forall n \geq n_1 : \quad 0 \leq f(n) \leq c_1 g(n).$$

We must show there exist positive c_2, n_2 s.t.

$$(2) \qquad \forall n \geq n_2 : \quad 0 \leq c_2 f(n) \leq g(n)$$

Let $c_2 = \frac{1}{c_1}$ and $n_2 = n_1$.

Then (1) $\Rightarrow$ (2), and proof is complete. ■

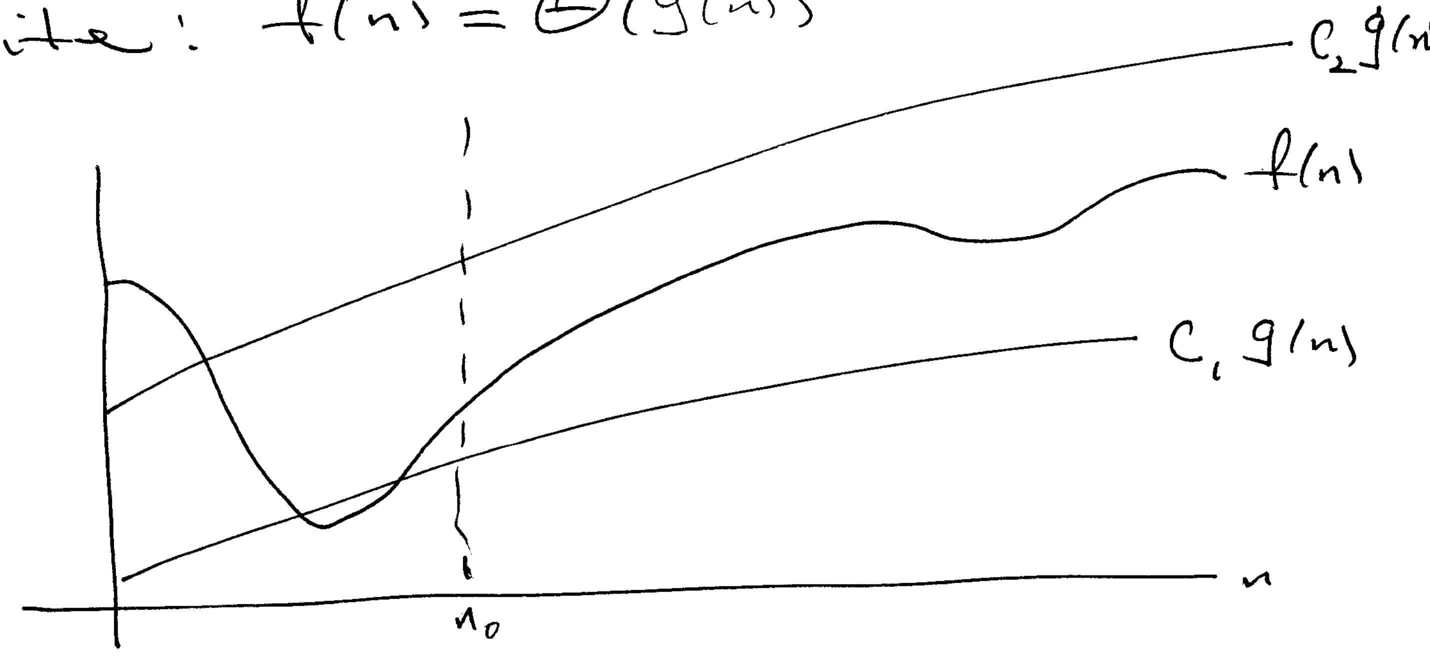
Defn

$$\Theta(g(n)) = \{ f(n) \mid \exists c_1 > 0, c_2 > 0, n_0 > 0, \forall n \geq n_0 : 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \}$$

we say: f(n) is big theta of g(n)

" " g(n) is a tight asym. bound for f(n)

write: $f(n) = \Theta(g(n))$



note: $\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$

equivalently (if f, g are a.p.)

$$\forall n \geq n_0 : \quad 0 \leq c_1 \leq \frac{f(n)}{g(n)} \leq c_2$$