CSE 201 1-21-21

Midterm 1 : Thur 1-28-21
see webpage for details

One more ind. handout example:

Ex. Show that $\forall n \geq 1$:

$P(n)$

if T is a tree with n vertices, then T has $n-1$ edges.

Proof:

I. $P(1)$ says: "if T is a tree with 1 vertex, then T has 0 edges"

note: only one graph with 1 vertex

so $P(1)$ is true.

II d. $\forall n > 1 : P(1) \wedge \dots \wedge P(n-1) \rightarrow P(n)$

Let $n > 1$ be arbitrary. Assume for this n, and for all k in the range $1 \le k < n$, that:

If T' is a tree with k vertices, then T' has $k-1$ edges.

We must show that:

If T is a tree with n vertices, then T has $n-1$ edges.

Assume T is a tree on n vertices. Let e be any edge in T and remove it. This decomposes T into two subtrees T_1, T_2, each with fewer than n vertices.

Let k_1, k_2 be # of vertices in T_1 and T_2, respectively. By the induction hypothesis:

T_1 has $k_1 - 1$ edges, and

T_2 has $k_2 - 1$ edges.

By re-inserting e into T, we see that the # edges in T is

$$(k_1 - 1) + (k_2 - \cancel{1}) + \cancel{1} = (k_1 + k_2) - 1$$

$$= n - 1$$

<u>note</u>: $k_1 + k_2 = n$ since no vertices were removed.

Thus T has $n-1$ edges, as required. Result follows for all trees by 2^{nd} PMI. ∎

Handout: Recurrence relations

Ex.

$$T(n) = \begin{cases} c & (n=1) \\ T(\lfloor \frac{n}{2} \rfloor) + T(\lceil \frac{n}{2} \rceil) + dn & (n \geq 2) \end{cases}$$

explicit solution: a fcn. that satisfies recurrence for all $n \geq 1$.

asymptotic solution: an asymptotic (upper, lower, tight) bound on the explicit solution.

3 methods

- Substitution: start with a guess as to asymp. (upper or lower) bound, then prove the guess by induction.

- Iteration/recursion tree: generate guess for substitution

- Master: use cookbook method to get a tight asymptotic bound.

Substitution method

Ex $T(n) = \begin{cases} 2 & 1 \le n < 3 \\ 3 \cdot T(\lfloor \frac{n}{3} \rfloor) + n & n \ge 3 \end{cases}$

Guess: $T(n) = O(n \log n)$.

To prove this, we must find positive c, n_0 such that

$$\forall n \ge n_0 : T(n) \le c n \log n$$

observe: recurrence has 2 base cases $n=1, n=2$. But base case $n=1$ cannot be true!

$$\underbrace{T(1)}_{2} \le \underbrace{c \cdot 1 \cdot \log(1)}_{0}$$

also

$$\underbrace{T(2)}_{2} \leq \underbrace{c \cdot 2 \log(2)}$$

can be made by proper choice of c.

lowest base case : $n_0 = 2$

highest base case : $n_1 = ?$

Lets assume $\log(\,) = \log_3(\,)$.

sketch of ind. proof.

Let $n > n_1$. assume for $2 \leq k < n$ that $T(k) \leq c\,k \log(k)$. we must show

$$T(n) \leq c\,n \log(n).$$

we have

$$T(n) = 3T(\lfloor \tfrac{n}{3} \rfloor) + n$$

$$\leq 3 \cdot c \left\lfloor \frac{n}{3} \right\rfloor \log\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n \quad \left\{ \begin{array}{l} \text{by ind. hyp.} \\ \text{with } k = \left\lfloor \frac{n}{3} \right\rfloor \end{array} \right.$$

$$\leq \cancel{3} c \left(\frac{n}{\cancel{3}}\right) \log\left(\frac{n}{3}\right) + n$$

$$= cn(\log n - 1) + n$$

$$= cn \log n - cn + n \underset{\substack{\uparrow \\ \text{want}}}{\leq} cn \log n$$

$\therefore$ we want $-cn + n \leq 0$

i.e. $n \leq cn$

i.e. $1 \leq c$

If this constraint is satisfied, then ind. step will work.

To apply the ind. hyp. we need

$$2 \leq \left\lfloor \frac{n}{3} \right\rfloor < n$$

obviously true

true only if $2 \le \frac{n}{3}$

i.e. $n \ge 6$ $\therefore$ pick $\boxed{n_1 = 5}$

Base cases: $n = 2, 3, 4, 5$

need

$$T(2) \le c \cdot 2 \log(2)$$

$$T(3) \le c \cdot 3 \log(3)$$

$$T(4) \le c\, 4 \log(4)$$

$$T(5) \le c\, 5 \log(5)$$

Pick

$$c = \max\left\{1, \frac{T(2)}{2\log 2}, \frac{T(3)}{3\log 3}, \frac{T(4)}{4\log 4}, \frac{T(5)}{5\log 5}\right\}$$

largest

pick $c = \frac{T(3)}{3\log 3} = \frac{9}{3\log 3} = \boxed{3}$

claim: $\forall n \geq 2$: $\underbrace{\boxed{T(n) \leq 3\, n \log_3(n)}}_{P(n)}$

proof:

I. 4 Base cases

$T(2) \leq 3 \cdot 2 \log 2$ ✓

$T(3) \leq 3 \cdot 3 \log 3$ ✓

$T(4) \leq 3 \cdot 4 \log 4$ ✓

$T(5) \leq 3 \cdot 5 \log 5$ ✓

II. $\forall n > 5$: $P(2) \wedge \ldots \wedge P(n-1) \longrightarrow P(n)$.

let $n > 5$ be arbitrary. assume for $2 \leq k < n$ that $T(k) \leq 3 k \log k$.
In particular, for $k = \lfloor \frac{n}{3} \rfloor$ we have

$$T(\lfloor \tfrac{n}{3} \rfloor) \leq 3 \cdot \lfloor \tfrac{n}{3} \rfloor \log \lfloor \tfrac{n}{3} \rfloor .$$

we must show $T(n) \le 3n\log n$

so

$$T(n) = 3T(\lfloor \tfrac{n}{3} \rfloor) + n$$

$$\le 3 \cdot 3 \lfloor \tfrac{n}{3} \rfloor \log \lfloor \tfrac{n}{3} \rfloor + n$$

$$\le 9 \cdot (\tfrac{n}{3}) \log (\tfrac{n}{3}) + n$$

$$= 3n(\log n - 1) + n$$

$$= 3n\log n - 3n + n$$

$$= 3n\log n - 2n$$

$$\le 3n\log n \,.$$

∎