CSE 201 1-19-21

Induction Proofs:

Goal: $\forall n \geq n_0 : P(n)$

I. Base step

Prove $P(n_0)$ directly.

IIa. Induction step

Prove: $\forall n \geq n_0 : \underbrace{P(n)}_{\substack{\text{induction}\\ \text{hypothesis}}} \rightarrow \underbrace{P(n+1)}_{\substack{\text{induction}\\ \text{conclusion}}}$.

Conclusion:

$\forall n \geq n_0 : P(n)$

II b. Prove

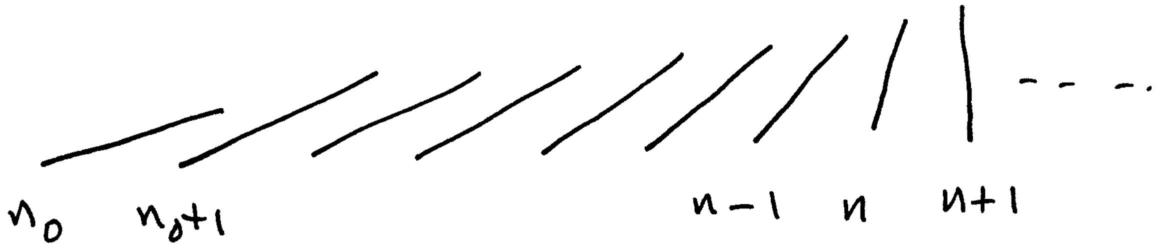
$$\forall n > n_0 : \underbrace{P(n-1)}_{\substack{\text{ind.}\\ \text{hyp.}}} \rightarrow \underbrace{P(n)}_{\substack{\text{ind.}\\ \text{conc.}}}$$

II a and II b are known as <u>weak induction</u> or the <u>1st Principle of mathematical induction</u> (<u>1st PMI</u>).

II c and II d are known as <u>strong induction</u> or as the <u>2nd PMI</u>.

IIc. Prove

$$\forall n \geq n_0 : \underbrace{P(n_0) \wedge P(n_0+1) \wedge \ldots \wedge P(n)}_{\text{ind. hyp.}} \rightarrow \underbrace{P(n+1)}_{\text{ind. conc.}}$$



IId. Prove

$$\forall n > n_0 : \overbrace{P(n_0) \wedge \ldots \wedge P(n-1)}^{\text{ind. hyp.}} \rightarrow \overbrace{P(n)}^{\text{ind. conc.}}$$

Ex. $\forall n \geq 1$: $\boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$ $\leftarrow P(n)$

Proof.

I. $P(1)$ says

$$\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2 \cdot 1+1)}{6}, \quad \text{i.e. } 1 = 1 \checkmark$$

II a. $\forall n \geq 1 : P(n) \rightarrow P(n+1)$

Let $n \geq 1$ be chosen arbitrarily.

Assume $P(n)$ is true, i.e. for this n

$$\left.\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.\right\}\ \text{explicit ind. hyp.}$$

we must show that $P(n+1)$ is also true, i.e

$$\left.\sum_{i=1}^{n+1} i^2 = \frac{(n+1)(n+2)(2n+3)}{6}\right\}\ \text{explicit ind. conclusion.}$$

so

$$\sum_{i=1}^{n+1} i^2 = \left(\sum_{i=1}^{n} i^2\right) + (n+1)^2$$

$$= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \ \left\{\ \text{By the induction hypothesis}\right.$$

$$= (n+1)\left(\frac{n(2n+1) + 6(n+1)}{6}\right)$$

$$= \frac{(n+1)(2n^2+n+6n+6)}{6}$$

$$= \frac{(n+1)(2n^2+7n+6)}{6}$$

$$= \frac{(n+1)(n+2)(2n+3)}{6}$$

By the 1^{st} PMI we conclude

$$\forall n \geq 1: \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

$\blacksquare$

<u>Exercise</u>: re-do this proof using ~~form~~ II b.

<u>Ex.</u> let $x \in \mathbb{R}$, $x \neq 1$.

$$\forall n \geq 0 \quad \boxed{\sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1}} \leftarrow P(n)$$

<u>Proof:</u>

$P(0)$ says $\displaystyle\sum_{i=0}^{0} x^i = \frac{x^{0+1} - 1}{x - 1}$

i.e. $x^0 = \dfrac{x-1}{x-1}$

i.e. $1 = 1$ ✓

II b. $\forall n > 0 : P(n-1) \to P(n)$

let $n > 0$ be arbitrary. Assume (for this n) that

$$\sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x - 1} . \qquad \text{(ind. hyp.)}$$

we must show that

$$\sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1} \qquad \text{(ind. conc.)}$$

so

$$\sum_{i=0}^{n} x^i = \left(\sum_{i=0}^{n-1} x^i \right) + x^n$$

$$= \left(\frac{x^n - 1}{x - 1} \right) + x^n \quad \Big\{ \begin{array}{l} \text{by the} \\ \text{ind. hyp.} \end{array}$$

$$= \frac{x^n - 1 + x^n(x-1)}{x-1}$$

$$= \frac{\cancel{x^n} - 1 + x^{n+1} - \cancel{x^n}}{x-1}$$

$$= \frac{x^{n+1} - 1}{x-1}.$$

The result follows by 1st PMI.

∎

Ex. Define $T(n)$ for $n \in \mathbb{Z}^+$ by

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1 & \text{if } n \geq 2 \end{cases}$$

Prove that $\forall n \geq 1$: $\boxed{T(n) \leq \lg(n)}$ $T(n)$.

$\log_2(n)$.

(Hence $T(n) = O(\log(n))$.)

Proof.

I. $P(1)$ says $T(1) \le \lg(1)$

i.e. $0 \le 0$ ✓

IId. $\forall n > 1 : \underbrace{P(1) \wedge P(2) \wedge \ldots \wedge P(n-1)} \Rightarrow P(n)$.

Let $n > 1$ be arbitrary. Assume $P(1) \wedge \ldots \wedge P(n-1)$ is true. i.e. we

$$\boxed{\text{assume for all } k \text{ in the range } 1 \le k < n \text{ that } T(k) \le \lg(k)}$$

explicit ind. hyp.

we must show that

$$\boxed{T(n) \le \lg(n)}$$

explicit ind. concl.

so

$$T(n) = T\left(\left\lfloor \tfrac{n}{2} \right\rfloor\right) + 1$$

$$\leq \lg\left(\left\lfloor \tfrac{n}{2} \right\rfloor\right) + 1$$

$\Big\{$ by the ind. hyp. with $k = \left\lfloor \frac{n}{2} \right\rfloor$. works since $1 \leq \left\lfloor \frac{n}{2} \right\rfloor < n$.

$$\leq \lg\left(\tfrac{n}{2}\right) + 1$$

$\Big\{$ since $\lfloor x \rfloor \leq x$.

$$= \lg(n) - \cancel{\lg(2)} + \cancel{1}$$

$$= \lg(n)$$

$\therefore$ $T(n) \leq \lg(n)$, result follows by 2nd PMI. ∎

Exercise define

$$S(n) = \begin{cases} 0 & \text{if } n=1 \\ S(\lceil n/2 \rceil)+1 & \text{if } n \geq 2. \end{cases}$$

show $\forall n \geq 1: S(n) \geq \lg(n)$.

(hence $S(n) = \Omega(\log n)$.)

multiple base cases

Base: $P(1), P(2), \ldots, P(n_0)$

strong ind. step: $\forall n > n_0 : P(1) \wedge \ldots \wedge P(n-1) \rightarrow P(n)$.

conclusion: $\forall n \geq 1 : P(n)$.

Ex. define

$$T(n) = \begin{cases} 1 & \text{if } 1 \le n \le 2 \\ 4T(\lfloor \frac{n}{3} \rfloor) + n & \text{if } n \ge 3 \end{cases}$$

show : $\forall n \ge 1 : \underbrace{\boxed{T(n) \le n^2}}_{P(n)}$ (hence $T(n) = O(n^2)$)

Proof:

I. 2 base cases : $n = 1$, $n = 2$.

$P(1)$ says $1 \le 1^2$ true ✓

$P(2)$ says $1 \le 2^2$ true ✓

II. $\forall n > 2 : P(1) \land P(2) \land \ldots \land P(n-1) \Rightarrow P(n)$.

let $n > 2$. Assume for all k in the range $1 \le k < n$ that

$$T(k) \le k^2.$$

we must show : $T(n) \le n^2$.

So

$$T(n) = 4T\left(\left\lfloor \tfrac{n}{3} \right\rfloor\right) + n$$

$$\le 4 \cdot \left\lfloor \tfrac{n}{3} \right\rfloor^2 + n$$ $\left\{\begin{array}{l}\text{by ind. hyp.} \\ \text{with } k = \lfloor \frac{n}{3} \rfloor. \\ \text{Valid since} \\ 1 \le \lfloor \frac{n}{3} \rfloor < n\end{array}\right.$

$$\le 4\left(\tfrac{n}{3}\right)^2 + n$$ $\left\{ \text{since } \lfloor x \rfloor \le x \right.$

$$= \tfrac{4}{9} n^2 + n \le n^2$$

$$\uparrow$$

$$n > 2 \Rightarrow \tfrac{9}{5} \le n \Rightarrow n \le \tfrac{5}{9} n^2$$

$$\Rightarrow \tfrac{4}{9} n^2 + n \le n^2 \quad \checkmark$$

$\therefore$ $T(n) \le n^2$, as required.

$\blacksquare$

hw2 #6 hint

1st PMI: for any $P(n)$

$$[P(1) \wedge (\forall n>1: P(n-1) \rightarrow P(n))] \rightarrow \forall n \geq 1: P(n)$$

2nd PMI: for any $Q(n)$

$$[Q(1) \wedge (\forall n>1: Q(1) \wedge \cdots \wedge Q(n-1) \rightarrow Q(n))] \rightarrow \forall n \geq 1: Q(n$$

Let $Q(n)$ be any prop. fcn. assume

(1) $Q(1)$
(2) $\forall n>1: Q(1) \wedge \cdots \wedge Q(n-1) \rightarrow Q(n)$
show
(3) $\forall n \geq 1: Q(n)$

Define $P(n)$ by

$P(n) =$ ... some logical exp. involving $Q(\cdot)$ ...

If you choose $P(\cdot)$ correctly, then

$P(1) = Q(1)$
$P(2) = \ ?$
$\vdots$
$P(n) = \ ?$

also $\forall n P(n) \Rightarrow \forall n Q(n)$ if P is correct.