handout on common fcns. (sec. 3.2)

Floor & ceiling fcns.

let $x \in \mathbb{R}$

(1) $\lfloor x \rfloor$ = greatest int. $\leq x$

(2) $\lceil x \rceil$ = least int $\geq x$

alternative definition ...

let $N \in \mathbb{Z}$, $x \in \mathbb{R}$.

(1) $N = \lfloor x \rfloor$ iff $N \leq x < N+1$

(2) $N = \lceil x \rceil$ iff $N-1 < x \leq N$

Lemma 1: let $x \in \mathbb{R}$, $a, b \in \mathbb{Z}$. Then

(1) $a \le x < b$ iff $a \le \lfloor x \rfloor < b$

(2) $a < x \le b$ iff $a < \lceil x \rceil \le b$

Lemma 2: if $x \in \mathbb{R}$, $m \in \mathbb{Z}^+$, then

(1) $\left\lfloor \frac{\lfloor x \rfloor}{m} \right\rfloor = \left\lfloor \frac{x}{m} \right\rfloor$

(2) $\left\lceil \frac{\lceil x \rceil}{m} \right\rceil = \left\lceil \frac{x}{m} \right\rceil$

Lemma 3: if $a, b \in \mathbb{Z}^+$, $n \in \mathbb{Z}^+$, then

(1) $\left\lfloor \frac{\lfloor n/a \rfloor}{b} \right\rfloor = \left\lfloor \frac{n}{ab} \right\rfloor$

(2) $\left\lceil \frac{\lceil n/a \rceil}{b} \right\rceil = \left\lceil \frac{n}{ab} \right\rceil$

see handout for proofs.

logs

let $x, a, b \in \mathbb{R}$ where $x > 0$, $a > 1$, $b > 1$.

$\log_a(x)$ is inverse of $\exp_a(x) = a^x$.

This means:

$$a^{\log_a x} = x$$

and

$$\log_a(a^x) = x$$

Therefore

$$(*) \qquad x = a^{\log_a x} = \left(b^{\log_b(a)}\right)^{\log_a x} = b^{\log_b(a) \cdot \log_a(x)}$$

Take $\log_b(\,)$ of both sides

$$(**) \qquad \log_b(x) = \underbrace{\log_b(a)}_{\text{const.}} \cdot \log_a(x)$$

hence

$$\log_b(n) = \text{const} \cdot \log_a(n)$$

$$\therefore \quad \boxed{\log_b(n) = \Theta(\log_a(n))}$$

conversion formula:

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)} \qquad (\text{by } **)$$

also by $(**)$ we have

$$a^{\log_b(x)} = a^{\log_b(a) \cdot \log_a(x)}$$

$$= \left(a^{\log_a(x)}\right)^{\log_b(a)}$$

$$= x^{\log_b(a)}$$

$$\therefore \quad \boxed{a^{\log_b(x)} = x^{\log_b(a)}}$$

Stirling's formula

Let $n \in \mathbb{Z}^+$. Then

$$\boxed{n! = \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)}$$

weaker version:

$$\boxed{n! \sim \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n}$$

Corollary

(1) $n! = o(n^n)$ ✓

(2) $n! = \omega(b^n)$ for any $b > 0$

(3) $\log(n!) = \Theta(n \log n)$

Proof of (1)

$$\frac{n!}{n^n} = \frac{\sqrt{2\pi n}\cdot\frac{\cancel{n^n}}{e^n}\cdot\left(1+\Theta\left(\frac{1}{n}\right)\right)}{\cancel{n^n}}$$

$$= \sqrt{2\pi}\cdot\frac{n^{1/2}}{e^n}\cdot\left(1+\Theta\left(\frac{1}{n}\right)\right) \longrightarrow 0$$

as $n \to \infty$

Exercise: Prove (2)

Proof of (3)

$$\log(n!) = \log\left(\sqrt{2\pi}\cdot n^{1/2}\cdot\frac{n^n}{e^n}\cdot\left(1+\Theta\left(\frac{1}{n}\right)\right)\right)$$

$$= \log(\sqrt{2\pi}) + \frac{1}{2}\log n + n\log n - n\log e$$

$$+\log\left(1+\Theta\left(\frac{1}{n}\right)\right)$$

$$\therefore \quad \frac{\log(n!)}{n\log n} = \underbrace{\frac{\log(\sqrt{2\pi})}{n\log n}}_{\downarrow\, 0} + \underbrace{\frac{1}{2n}}_{\downarrow\, 0} + \underbrace{1}_{\downarrow\, 1} - \underbrace{\frac{\log e}{\log n}}_{\downarrow\, 0} + \underbrace{\frac{\log(\qquad)}{n\log n}}_{\downarrow\, 0}$$

$$\therefore \lim_{n \to \infty} \left(\frac{\log n!}{n \log n} \right) = 1$$

$$\therefore \log n! = \Theta(n \log n). \quad \blacksquare$$

<u>Ex.</u> $\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$

Recall: $\binom{m}{k} = \dfrac{m!}{k!\,(m-k)!}$

<u>Proof:</u>

$$\binom{2n}{n} = \frac{(2n)!}{n!\,(2n-n)!}$$

$$= \frac{(2n)!}{(n!)^2}$$

$$= \frac{\sqrt{\cancel{2}\pi\cdot\cancel{2}n}\cdot\left(\frac{2n}{e}\right)^{2n}\cdot\left(1+\Theta\left(\frac{1}{2n}\right)\right)}{\cancel{2}\pi n\cdot\left(\frac{n}{e}\right)^{2n}\cdot\left(1+\Theta\left(\frac{1}{n}\right)\right)^2}$$

$$= \frac{1}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{n}}\cdot\frac{2^{2n}\cdot\cancel{n^{2n}}}{\cancel{e^{2n}}}\cdot\frac{\cancel{e^{2n}}}{\cancel{n^{2n}}}\cdot\left[\frac{\left(1+\Theta\left(\frac{1}{2n}\right)\right)}{\left(1+\Theta\left(\frac{1}{n}\right)\right)^2}\right]$$

$$= \frac{1}{\sqrt{\pi}}\cdot\frac{4^n}{\sqrt{n}}\cdot\left[\frac{\cdots}{\cdots}\right]$$

$$\therefore \frac{\binom{2n}{n}}{\frac{4^n}{\sqrt{n}}} = \frac{1}{\sqrt{\pi}}\cdot\left[\frac{\cdots}{\cdots}\right] \longrightarrow \frac{1}{\sqrt{\pi}} \text{ as } n\to\infty$$

since $0 < \frac{1}{\sqrt{\pi}} < \infty$, we have

$$\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right).$$

$\blacksquare$

Pascal's Δ

$$
\begin{array}{ccccccccccccc}
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 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 2 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & & & & & \binom{2n}{n} & & & & & & \\
 & & & & & & \vdots & & & & & &
\end{array}
$$

Exercise.

determine $a > 0$ such that

$$\binom{3n}{n} = \Theta\left(\frac{a^n}{\sqrt{n}}\right)$$

Exercise

for each $k \geq 2$, determine $a(k)$ such that

$$\binom{kn}{n} = \Theta\left(\frac{a(k)^n}{\sqrt{n}}\right)$$

Know: $a(2) = 4$

$a(3) =$ prev. exercise.

$\vdots$

Induction handout

Let $P(n)$ be a propositional function

$$P(n) = \begin{pmatrix} \text{a sentence whose truth or falsity} \\ \text{depends on } n \in \mathbb{N} = \{0, 1, 2, \dots\} \end{pmatrix}$$

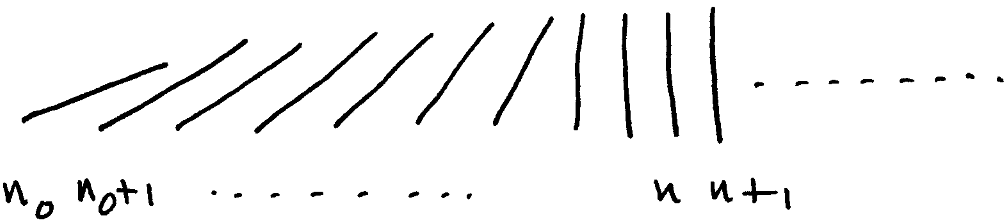
in a way

$$P : \mathbb{N} \rightarrow \{\text{false}, \text{true}\}$$

Goal: Prove ~~statements~~ of the form

$$\forall n \geq n_0 : P(n)$$

Domino analogy:



$P(n)$ = "the n^{th} domino falls".

Goal: $\forall n \geq n_0 : P(n)$

= "all dominos (after n_0) fall".

<u>Induction Proof.</u>

<u>analogy</u>

I. <u>Base step</u>: Prove $P(n_0)$ } "domino n_0 falls"

IIa <u>Induction step</u>:

$\forall n \geq n_0 : P(n) \rightarrow P(n+1)$ } "if any domino falls, then the next also falls"

once I and IIa are proven,
we may conclude

$$\forall n \geq n_0 : P(n)$$ "all dominos fall"

continue .....