1

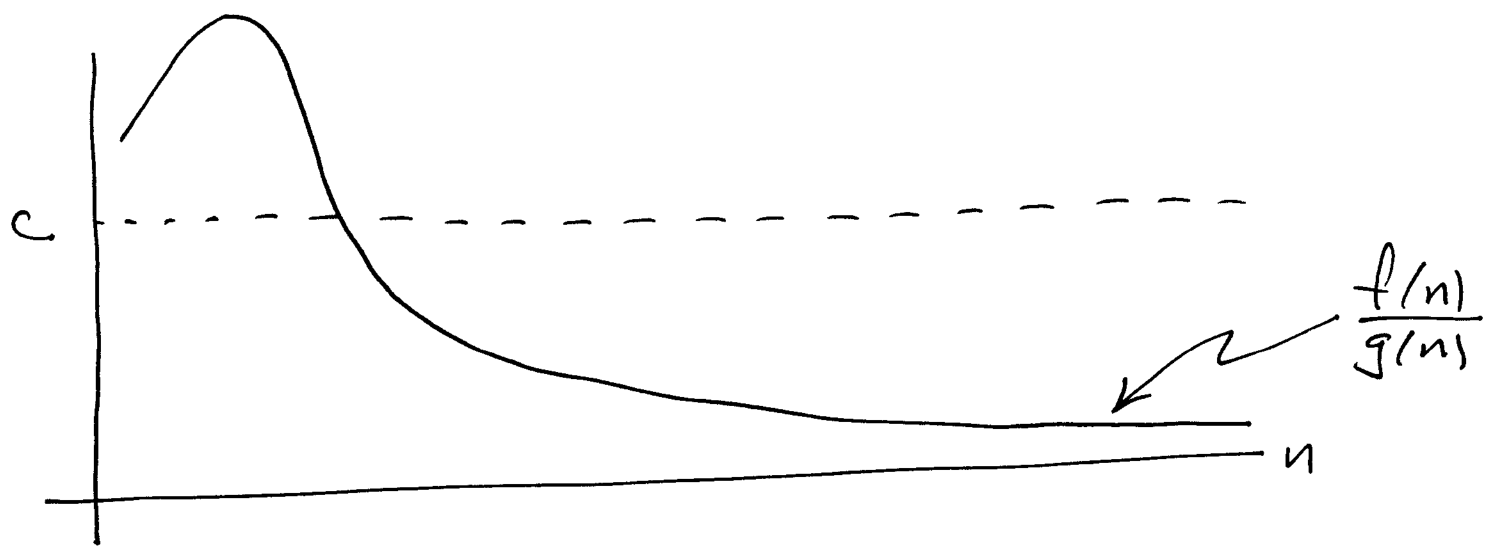
cse 201 1-12-21

continue ....

defn. of $o(g(n))$

many ~~texts~~ ~~take~~ limit as defn.

$f(n) = o(g(n))$ iff $\frac{f(n)}{g(n)} \to 0$ as $n \to \infty$.



also the example

$$n^k = o(e^n)$$

generalizes to

$$n^k = o(b^n) \quad \text{for any } b>1$$

Proof: exercise

Question: how many app. of l'hop?

Answer: $\lceil k \rceil$ many app. of

we now know

$$\text{all logs} < \text{all poly} < \text{all exp}$$

means $o(\,)$ here

Defn

$$\omega(g(n)) = \{ f(n) \mid \forall c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq c\,g(n) < f(n) \}$$

write $f(n) = \omega(g(n))$

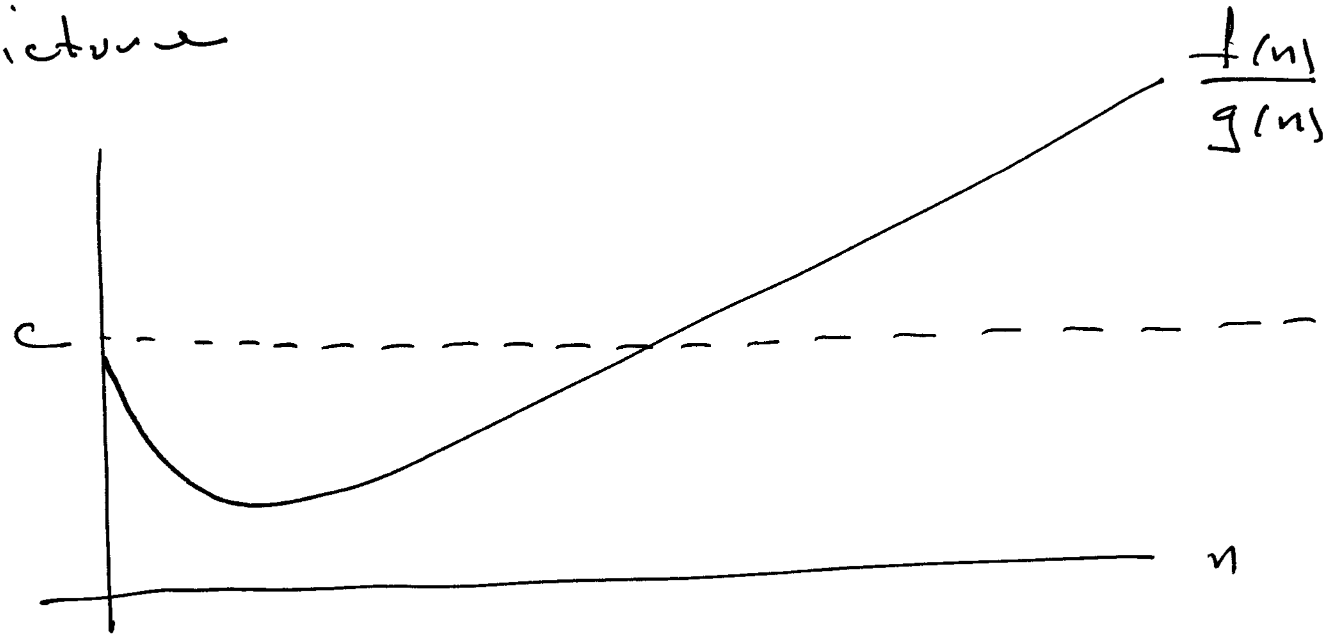
say $g(n)$ is a strict asym. l.b. for $f(n)$

Recall Ω

$$\Omega(g(n)) = \{ f(n) \mid \exists c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq c\,g(n) \leq f(n) \}$$

observe : $\omega(g(n)) \subseteq \Omega(g(n))$

Picture



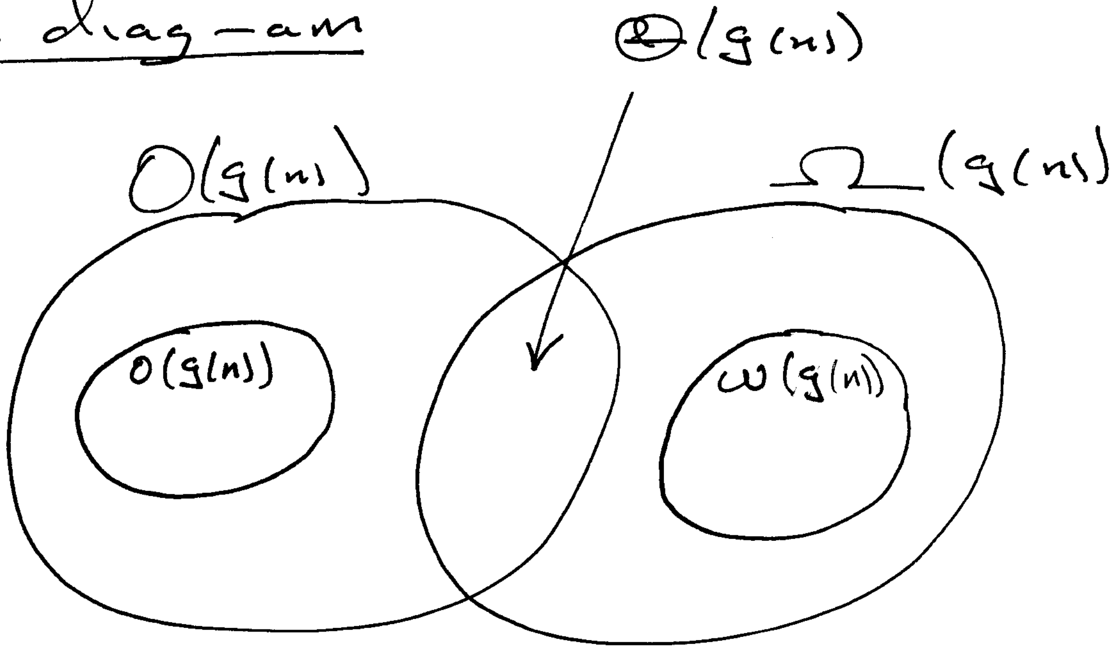
Lemma $f(n) = \omega(g(n))$ iff $\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = \infty$

Proof: exercise.

Exercise.

- Prove $o(g(n)) \cap \Omega(g(n)) = \phi$.
- Prove $\omega(g(n)) \cap O(g(n)) = \phi$.

Venn diagram



Theorem

If $\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = L$ where $0 \le L < \infty$,

then $f(n) = O(g(n))$.

warning: converse is false!

Proof.

limit statement says:

$$\forall \varepsilon > 0, \exists n_0 > 0, \forall n \ge n_0 : \left| \frac{f(n)}{g(n)} - L \right| < \varepsilon$$

since this holds for all $\varepsilon > 0$, we can choose $\varepsilon = 1$. Then there exists pos. n_0 st. for all $n \ge n_0$:

$$\left| \frac{f(n)}{g(n)} - L \right| < 1$$

$$\therefore \quad -1 < \frac{f(n)}{g(n)} - L < 1$$

this implies

$$\frac{f(n)}{g(n)} < L+1$$

$\therefore \quad f(n) < (L+1)\cdot g(n)$

$\therefore \; \exists \; c>0$ (namely $c = L+1$) and

$\exists \; n_0 > 0$ s.t. $\forall n \geq n_0$

$$0 \leq f(n) \leq c\, g(n)$$

recall blanket assumption: all fcns. under disc. are a.n.n.

$\therefore \quad f(n) = O(g(n)).$ $\blacksquare$

<u>Theorem</u>

If $\lim\limits_{n\to\infty} \left(\frac{f(n)}{g(n)}\right) = L$ where $0 < L \leq \infty$,

then $f(n) = \Omega(g(n))$.

<u>note</u>: converse is false.

Proof:

The limit implies

$$\lim_{n\to\infty}\left(\frac{g(n)}{f(n)}\right) = L' \quad \text{where } L' = \frac{1}{L}$$

i.e. $0 \le L' < \infty$. By Prev. Theorem we have

$$g(n) = O(f(n)).$$

By another thm, this implies

$$f(n) = \Omega(g(n)). \qquad \blacksquare$$

Thm

If $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = L$, where $0 < L < \infty$,

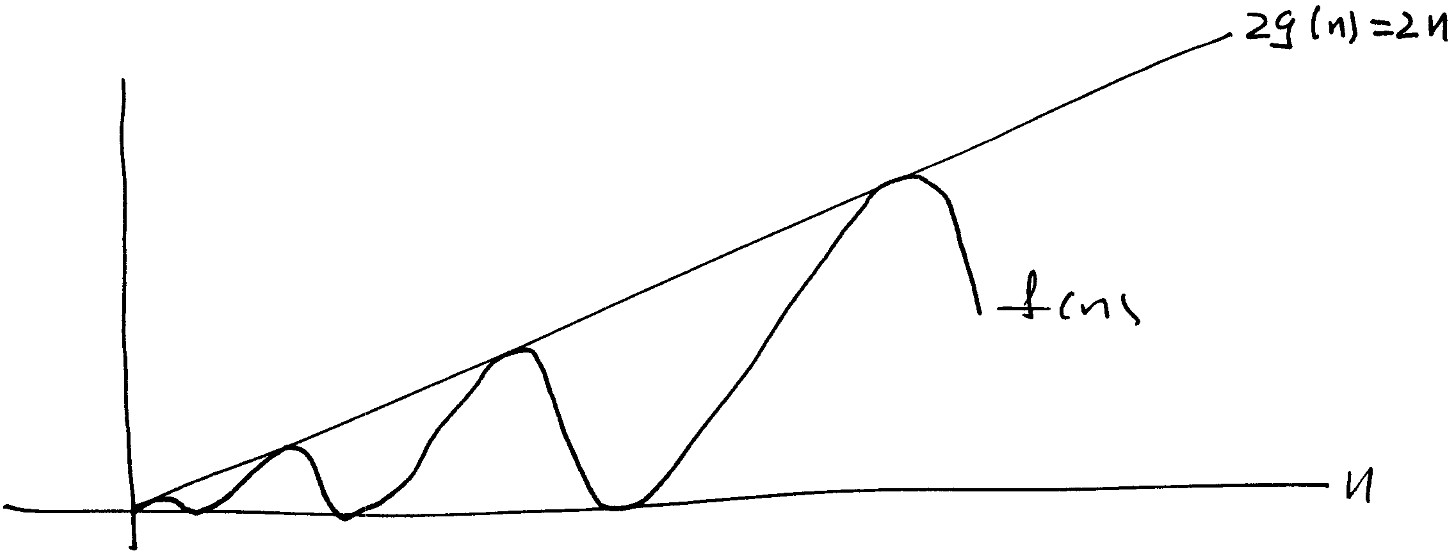
then $f(n) = \Theta(g(n))$.

Proof: exercise.

again: converse is false!

Example A.

$g(n) = n$, $f(n) = (1+\sin(n))n$



obviously $f(n) = O(g(n))$.

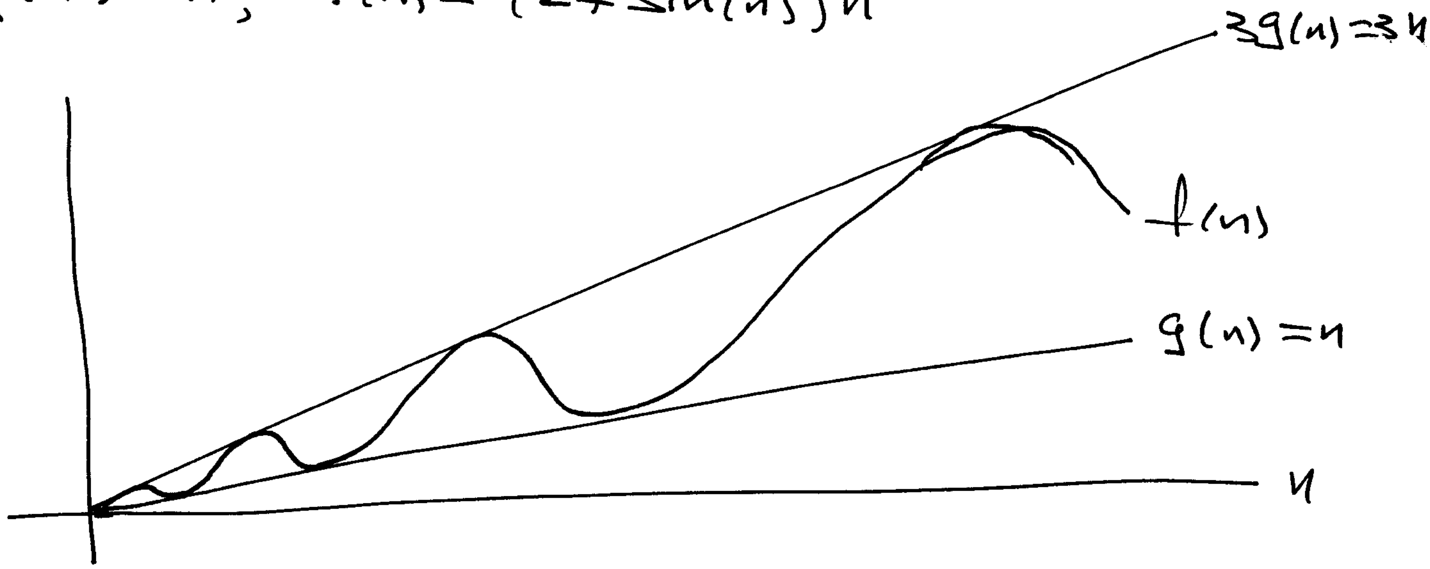
also $f(n) \neq \Omega(g(n))$.

however

$$\frac{f(n)}{g(n)} = 1+\sin(n) \quad \text{has no limit}$$

Example B

$g(n) = n, \quad f(n) = (2 + \sin(n))n$



obviously $f(n) = \Theta(g(n))$.

however

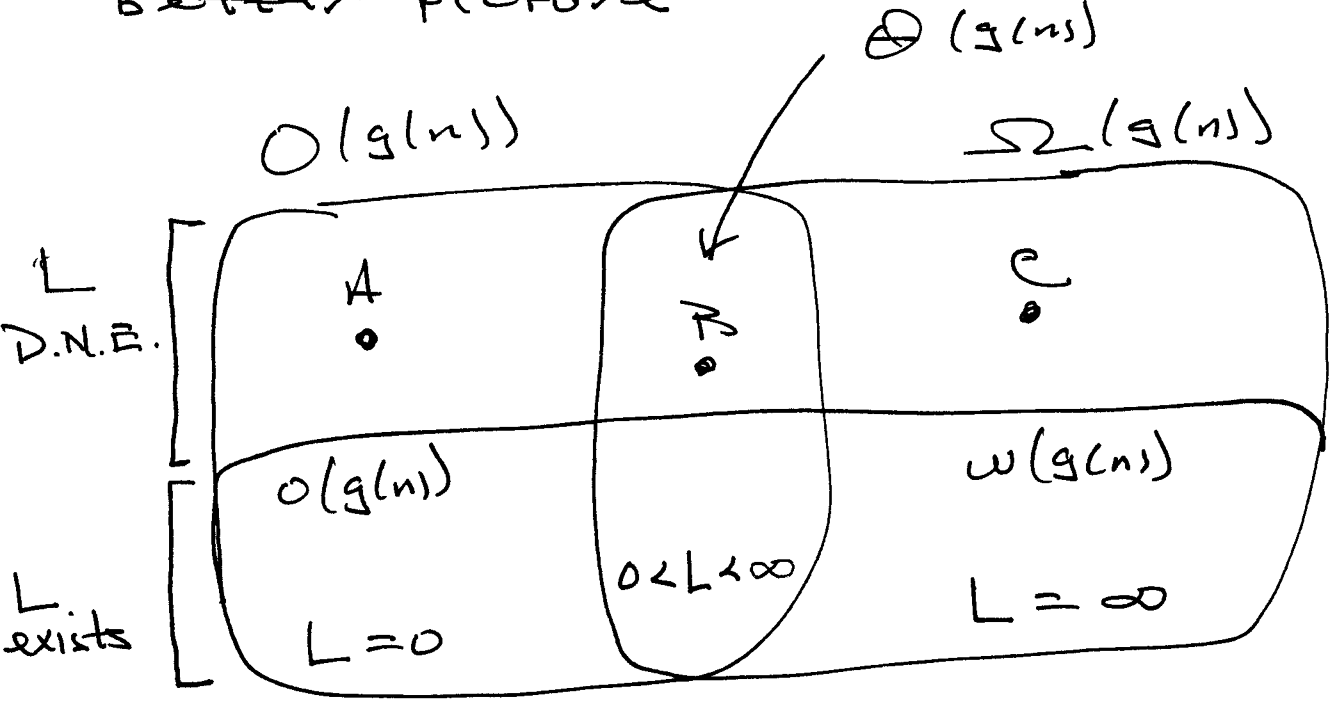
$$\frac{f(n)}{g(n)} = 2 + \sin(n) \quad \text{has no limit.}$$

Exercise: (Example C)

find fcns $g(n)$, $f(n)$ s.t.

- $f(n) = \Omega(g(n))$
- $f(n) \neq O(g(n))$
- $\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right)$ does not exist

better picture



let $L = \lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right)$ if it exists.

<u>Example</u>: $(n+a)^b = \Theta(n^b)$ for $b>0$.

$$\lim_{n\to\infty} \frac{(n+a)^b}{n^b} = \lim_{n\to\infty}\left(\frac{n+a}{n}\right)^b$$

$$= \left(\lim_{n\to\infty}\left(1+\frac{a}{n}\right)\right)^b$$

$$= 1^b = 1$$

and $0<1<\infty$.

Exercise (bottom of p. 7)

e.) if $P(n)$ is a poly of deg k, then

$P(n) = \Theta(n^k)$.

Proof:

$$P(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_2 n^2 + a_1 n + a_0$$

$$\therefore \frac{P(n)}{n^k} = a_k + \underbrace{\frac{a_{k-1}}{n}}_{\downarrow \atop 0} + \cdots\cdots + \underbrace{\frac{a_2}{n^{k-2}}}_{\downarrow \atop 0} + \underbrace{\frac{a_1}{n^{k-1}}}_{\downarrow \atop 0} + \underbrace{\frac{a_0}{n^k}}_{\downarrow \atop 0}$$

$$\therefore \lim_{n\to\infty} \left(\frac{P(n)}{n^k}\right) = a_k > 0.$$

$\uparrow$ all fcns are a.n.n.

$$\therefore \quad 0 < a_k < \infty$$

$$\therefore \quad P(n) = \Theta(n^k).$$

$\blacksquare$

d.) $$n^\alpha = \begin{cases} o(n^\beta) & \alpha < \beta \\ \Theta(n^\beta) & \alpha = \beta \\ \omega(n^\beta) & \alpha > \beta \end{cases}$$

Proof

$$\frac{n^\alpha}{n^\beta} = n^{\alpha-\beta} \longrightarrow \begin{cases} 0 & \text{if } \alpha < \beta \\ 1 & \text{if } \alpha = \beta \\ \infty & \text{if } \alpha > \beta \end{cases}$$

follows from limit thm. $\blacksquare$

e.) $$a^n = \begin{cases} o(b^n) & \text{if } a < b \\ \Theta(b^n) & \text{if } a = b \\ \omega(b^n) & \text{if } a > b \end{cases}$$

Proof

$$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n \longrightarrow \begin{cases} 0 & \text{if } a < b \\ 1 & \text{if } a = b \\ \infty & \text{if } a > b \end{cases}$$

follows from limit thm.

9.) $f(n) + o(f(n)) = \Theta(f(n))$

Proof.

let $h(n) = o(f(n))$. Then

$$\frac{h(n)}{f(n)} \longrightarrow 0 \quad \text{as} \quad n \to \infty$$

$$\therefore \quad \frac{f(n)+h(n)}{f(n)} = 1 + \underbrace{\frac{h(n)}{f(n)}}_{\downarrow\, 0} \longrightarrow 1$$

since $0 < 1 < \infty$, we have

$$f(n) + h(n) = \Theta(f(n)) \qquad \blacksquare$$

<u>Analogy:</u>

$f(n) = O(g(n)) \longleftrightarrow x \leq y$

$\Omega \longleftrightarrow \geq$

$\Theta \longleftrightarrow =$

$o \longleftrightarrow <$

$\omega \longleftrightarrow >$

<u>Exercise (bottom of p. 8)</u>

c.) compare 3^{2^n} to 2^{3^n}

$$\lim_{n\to\infty}\left(\frac{3^{2^n}}{2^{3^n}}\right) = 0 \quad \therefore \quad \boxed{3^{2^n} = o(2^{3^n})}$$

$$\lim_{n\to\infty}\left(\ln\left(\frac{3^{2^n}}{2^{3^n}}\right)\right) = -\infty$$

$$\ln\left(\frac{3^{2^n}}{2^{3^n}}\right) = \ln\left(3^{2^n}\right) - \ln\left(2^{3^n}\right)$$

$$= 2^n(\ln 3) - 3^n(\ln 2) \longrightarrow -\infty$$

$$= 3^n(\ln 2)\Big(\underbrace{\left(\tfrac{2}{3}\right)^n}_{\downarrow\, 0}\left(\frac{\ln 3}{\ln 2}\right) - 1\Big) \longrightarrow -\infty$$

<u>Defn</u> $f(n) \sim g(n)$ iff $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = 1$

<u>Exercise</u> show

$f(n) \sim g(n)$ iff $f(n) = g(n) + o(g(n))$

<u>note</u>: $f(n) = g(n) + o(g(n))$

iff $f(n) - g(n) = o(g(n))$

iff $\dfrac{f(n) - g(n)}{g(n)} \longrightarrow 0$

iff $\dfrac{f(n)}{g(n)} - 1 \longrightarrow 0$

iff $\dfrac{f(n)}{g(n)} \longrightarrow 1$ iff. $f(n) \sim g(n)$ ∎