

CSE 201
Homework Assignment 1
Solutions

1. (Problem 3.1-1) Let $f(n)$ and $g(n)$ asymptotically positive functions. Prove that $f(n) + g(n) = \Theta(\max(f(n), g(n)))$.

Proof:

Since $f(n)$ and $g(n)$ are asymptotically positive, we know that there exists a positive constant n_0 such that $f(n) > 0$ and $g(n) > 0$ for all $n \geq n_0$. For any such n we have

$$\begin{aligned} 0 &\leq \max(f(n), g(n)) \\ &\leq \min(f(n), g(n)) + \max(f(n), g(n)) \\ &\leq 2 \cdot \max(f(n), g(n)). \end{aligned}$$

But $f(n) + g(n) = \min(f(n), g(n)) + \max(f(n), g(n))$, and therefore

$$0 \leq 1 \cdot \max(f(n), g(n)) \leq f(n) + g(n) \leq 2 \cdot \max(f(n), g(n))$$

for all $n \geq n_0$, showing that $f(n) + g(n) = \Theta(\max(f(n), g(n)))$. ■

2. Prove or disprove: If $f(n) = \Theta(g(n))$, then $f(n)^2 = \Theta(g(n)^2)$.

Proof:

Since $f(n) = \Theta(g(n))$, there are positive constants c_1 , c_2 and n_0 such that

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$$

for all $n \geq n_0$. Squaring this inequality gives $0 \leq c_1^2 g(n)^2 \leq f(n)^2 \leq c_2^2 g(n)^2$, showing that $f(n)^2 = \Theta(g(n)^2)$. ■

3. Prove or disprove: If $f(n) = \Theta(g(n))$, then $2^{f(n)} = \Theta(2^{g(n)})$.

Counter Example:

Let $f(n) = 2n$ and $g(n) = n$. Then $f(n) = \Theta(g(n))$, but $2^{2n} = 4^n = \omega(2^n)$, and therefore $2^{f(n)} = \omega(2^{g(n)})$, whence $2^{f(n)} \neq \Theta(2^{g(n)})$. ■

4. Let $f(n)$ and $g(n)$ be asymptotically positive functions, and assume that $\lim_{n \rightarrow \infty} g(n) = \infty$. Prove that if $f(n) = \Theta(g(n))$, then $\ln(f(n)) = \Theta(\ln(g(n)))$.

Proof:

Assume $f(n) = \Theta(g(n))$. Then there exist positive constants c_1 , c_2 and n_0 such that

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$$

for all $n \geq n_0$. Since $\lim_{n \rightarrow \infty} g(n) = \infty$, the constant n_0 can be chosen large enough so that

$$1 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$$

for all $n \geq n_0$. Take $\ln()$ of all terms in the preceding inequality to get

$$0 \leq \ln(c_1) + \ln(g(n)) \leq \ln(f(n)) \leq \ln(c_2) + \ln(g(n))$$

for all $n \geq n_0$. Since $\lim_{n \rightarrow \infty} g(n) = \infty$, the term $\ln(g(n))$ dominates the constants $\ln(c_1)$ and $\ln(c_2)$, whence $\ln(c_1) + \ln(g(n)) = \Omega(\ln(g(n)))$ and $\ln(c_2) + \ln(g(n)) = O(\ln(g(n)))$. The desired result $\ln(f(n)) = \Theta(\ln(g(n)))$ now follows by Exercise 4 on page 4 of the handout on asymptotic growth rates. ■

5. (Problem 3.2-8) Show that if $f(n) \ln f(n) = \Theta(n)$, then $f(n) = \Theta(n/\ln n)$. Hint: use the result of the preceding problem.

Proof:

Assume $f(n) \ln f(n) = \Theta(n)$. Since $\lim_{n \rightarrow \infty} n = \infty$, we can, by the result of problem (4), take $\ln()$ of both sides of this equation to get

$$\ln(f(n)) + \ln(\ln(f(n))) = \Theta(\ln(n)).$$

Since the term $\ln(\ln(f(n))) = o(\ln(f(n)))$, we can, by Exercise 9g on page 8 of the handout on asymptotic growth rates, drop the lower order term and write this as $\ln(f(n)) = \Theta(\ln(n))$. Substituting this into our assumption gives $f(n)\Theta(\ln n) = \Theta(n)$, and therefore

$$f(n) = \Theta(n) \cdot \frac{1}{\Theta(\ln n)} = \Theta(n) \cdot \Theta\left(\frac{1}{\ln n}\right) = \Theta\left(\frac{n}{\ln n}\right)$$

where we have used Exercise 11bc on page 8 of the handout on asymptotic growth rates. ■

6. List the following 27 functions from lowest to highest asymptotic order. Indicate whether any two (or more) are of the same asymptotic order. Justify your answers.

$$2^{\lg(n)}, (\lg n)^{\lg n}, n^{1/\lg n}, \ln(\ln(n)), 4^{\lg n}, \sqrt{2}^{\lg n}, n, \sqrt{2n}, 1 \\ n^2, n!, (3/2)^n, n^3, (\lg n)^2, 2^{2^n}, n2^n, n^{\lg(\lg n)}, \ln(n), e^n \\ 2^{\sqrt{2} \lg n}, 2^n, n \lg n, 2^{2^{n+1}}, \lg(n!), \sqrt{\lg n}, 2^{(\lg n)^2}, (n+1)!$$

Solution:

The functions listed in increasing order are:

- (1) $1, n^{1/\lg n}$
- (2) $\ln(\ln(n))$
- (3) $\sqrt{\lg n}$
- (4) $\ln(n)$

- (5) $(\lg n)^2$
- (6) $2^{\sqrt{2 \lg n}}$
- (7) $\sqrt{2n}, \sqrt{2}^{\lg n}$
- (8) $n, 2^{\lg(n)}$
- (9) $n \lg n, \lg(n!)$
- (10) $n^2, 4^{\lg n}$
- (11) n^3
- (12) $n^{\lg(\lg n)}, (\lg n)^{\lg n}$
- (13) $2^{(\lg n)^2}$
- (14) $(3/2)^n$
- (15) 2^n
- (16) $n2^n$
- (17) e^n
- (18) $n!$
- (19) $(n+1)!$
- (20) 2^{2^n}
- (21) $2^{2^{n+1}}$

Justifications:

- (1) Let $y = n^{1/\lg n}$, then $\lg y = \lg(n^{1/\lg n}) = (1/\lg n) \cdot \lg n = 1$, hence $y = 2$ and $n^{1/\lg n} = 2 = \Theta(1)$.
- (2) $\lim_{n \rightarrow \infty} (\ln(\ln n)/1) = \infty$, so that $\ln(\ln n) = \omega(1)$.
- (3) Let $y_n = \sqrt{\lg n} / \ln(\ln n)$. Then $\lg y_n = \frac{1}{2} \lg(\lg n) - \lg(\ln(\ln n)) \rightarrow \infty$ as $n \rightarrow \infty$, hence $y_n \rightarrow \infty$ as $n \rightarrow \infty$, and therefore $\sqrt{\lg n} = \omega(\ln(\ln n))$.
- (4) Observe $\frac{\ln n}{\sqrt{\lg n}} = \frac{c \lg n}{\sqrt{\lg n}} = c \cdot \sqrt{\lg n} \rightarrow \infty$, where c is a constant, so $\ln n = \omega(\sqrt{\lg n})$.
- (5) We have $(\lg n)^2 / \ln n = (\lg n)^2 / c \lg n = \lg n / c \rightarrow \infty$, and hence $(\lg n)^2 = \omega(\ln n)$.
- (6) Let $y_n = 2^{\sqrt{2 \lg n}} / (\lg n)^2$. Then $\lg y_n = \sqrt{2} \cdot \sqrt{\lg n} - 2 \lg(\lg n) \rightarrow \infty$, showing $y_n \rightarrow \infty$, and therefore $2^{\sqrt{2 \lg n}} = \omega((\lg n)^2)$.
- (7) Observe that $\sqrt{2}^{\lg n} = n^{\lg \sqrt{2}} = n^{1/2} = \sqrt{n} = \Theta(\sqrt{2n})$. Now let $y_n = \sqrt{n} / 2^{\sqrt{2 \lg n}}$. Then $\lg y_n = \frac{1}{2} \lg n - \sqrt{2} \cdot \sqrt{\lg n} \rightarrow \infty$, so $y_n \rightarrow \infty$, and $\sqrt{n} = \omega(2^{\sqrt{2 \lg n}})$.
- (8) Note $2^{\lg n} = n^{\lg 2} = n = \omega(\sqrt{n})$.
- (9) Recall we showed $\lg(n!) = \Theta(n \lg n)$ using Stirling's formula. Also $n \lg n / n = \lg n \rightarrow \infty$, so that $n \lg n = \omega(n)$.
- (10) We have $4^{\lg n} = n^{\lg 4} = n^2$, and $n^2 / n \lg n = n / \lg n \rightarrow \infty$ (by l'Hopital's rule), hence $n^2 = \omega(n \lg n)$.
- (11) Clearly $n^3 = \omega(n^2)$, since $n^3 / n^2 = n \rightarrow \infty$.
- (12) Observe $(\lg n)^{\lg n} = n^{\lg(\lg n)}$. Note $n^{\lg(\lg n)} / n^3 = n^{\lg(\lg n) - 3} \rightarrow \infty$, since $(\lg(\lg n) - 3) \rightarrow \infty$, showing that $n^{\lg(\lg n)} = \omega(n^3)$.
- (13) Let $y_n = 2^{(\lg n)^2} / n^{\lg(\lg n)}$. Then $\lg y_n = (\lg n)^2 - (\lg \lg n) \cdot (\lg n) \rightarrow \infty$, whence $y_n \rightarrow \infty$, and therefore $2^{(\lg n)^2} = \omega(n^{\lg(\lg n)})$.
- (14) Let $y_n = (3/2)^n / 2^{(\lg n)^2}$. Then $\lg y_n = n \lg(3/2) - (\lg n)^2 \rightarrow \infty$, so that $y_n \rightarrow \infty$, and hence $(3/2)^n = \omega(2^{(\lg n)^2})$.
- (15) We have $2^n = \omega((3/2)^n)$, since $2 > 3/2$.

- (16) $n2^n = \omega(2^n)$ since $n2^n/2^n = n \rightarrow \infty$.
- (17) Observe $e^n = \omega(n2^n)$, since $\lim_{n \rightarrow \infty} \frac{e^n}{n2^n} = \lim_{n \rightarrow \infty} \frac{(e/2)^n}{n} = \lim_{n \rightarrow \infty} \frac{\text{const} \cdot (e/2)^n}{1} = \infty$, using l'Hopital's rule and the fact that $e > 2$.
- (18) Let $y_n = n!/e^n$. Then $\lg y_n = \lg(n!) - n \lg e = \theta(n \lg n) - \theta(n) \rightarrow \infty$, whence $y_n \rightarrow \infty$, and therefore $n! = \omega(e^n)$.
- (19) We have $(n+1)!/n! = (n+1) \rightarrow \infty$, so that $(n+1)! = \omega(n!)$.
- (20) Let $y_n = 2^{2^n}/(n+1)!$. Then $\lg y_n = 2^n - \lg((n+1)!) = 2^n - \theta((n+1) \lg(n+1)) = 2^n - \theta(n \lg n)$, therefore $\lg y_n \rightarrow \infty$, hence $y_n \rightarrow \infty$, showing that $2^{2^n} = \omega((n+1)!)$.
- (21) Observe $2^{n+1} - 2^n = 2^n(2-1) = 2^n \rightarrow \infty$, so that $2^{2^{n+1}}/2^{2^n} = 2^{2^{n+1}-2^n} \rightarrow \infty$, showing that $2^{2^{n+1}} = \omega(2^{2^n})$. ■

7. Determine the asymptotic order of the expression $\sum_{i=1}^n a^i$ where $a > 0$ is a constant, i.e. find a simple function $g(n)$ such that the expression is in the class $\Theta(g(n))$. (Hint: consider the cases $a = 1$, $a > 1$, and $0 < a < 1$ separately.)

Solution: In the case $a = 1$ we have $\sum_{i=1}^n a^i = n = \Theta(n)$. Now assume $a \neq 1$ and use the formula for the sum of a geometric series:

$$\sum_{i=1}^n a^i = a \left(\sum_{i=0}^{n-1} a^i \right) = a \left(\frac{a^n - 1}{a - 1} \right) = \left(\frac{a}{a-1} \right) \cdot a^n - \left(\frac{a}{a-1} \right)$$

If $a > 1$ then $a^n \rightarrow \infty$ as $n \rightarrow \infty$, so the first term dominates the second, and $\sum_{i=1}^n a^i = \Theta(a^n)$. If $0 < a < 1$ then $a^n \rightarrow 0$, so the constant term (which is positive in this case) dominates, and therefore $\sum_{i=1}^n a^i = \Theta(1)$. ■

8. Use induction to prove that $\sum_{k=1}^n k^4 = \frac{n(n+1)(6n^3+9n^2+n-1)}{30}$ for all $n \geq 1$.

Proof:

I. $\sum_{k=1}^1 k^4 = 1 = \frac{2 \cdot 15}{30} = \frac{1 \cdot (1+1) \cdot (6 \cdot 1^3 + 9 \cdot 1^2 + 1 - 1)}{30}$, and the base case is established.

II. Let $n \geq 1$. Assume for this n that

$$\sum_{k=1}^n k^4 = \frac{n(n+1)(6n^3+9n^2+n-1)}{30}$$

We must show that

$$\sum_{k=1}^{n+1} k^4 = \frac{(n+1)(n+2)(6(n+1)^3+9(n+1)^2+(n+1)-1)}{30}$$

We have

$$\sum_{k=1}^{n+1} k^4 = \left(\sum_{k=1}^n k^4 \right) + (n+1)^4$$

$$\begin{aligned}
&= \frac{n(n+1)(6n^3+9n^2+n-1)}{30} + (n+1)^4 && \text{(by the induction hypothesis)} \\
&= \frac{(n+1)[n(6n^3+9n^2+n-1)+30(n+1)^3]}{30} \\
&= \frac{(n+1)[6n^4+39n^3+91n^2+89n+30]}{30} \\
&= \frac{(n+1)(n+2)[6n^3+27n^2+37n+15]}{30} \\
&= \frac{(n+1)((n+1)+1)[6(n+1)^3+9(n+1)^2+(n+1)-1]}{30}
\end{aligned}$$

Thus the formula holds for $n + 1$ as well. The formula is valid for all n by induction. ■