CSE 201 1-9-20

Exercise

let $h_1(n) = \Omega(g(n))$, $h_2(n) = O(g(n))$

and suppose

$$h_1(n) \le f(n) \le h_2(n)$$

for all suff. large n. Then

$$f(n) = \Theta(g(n)).$$

Ex. let $k \in \mathbb{R}$ with $k \geq 0$.

show $\displaystyle\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$

note:

- $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \Theta(n^2)$
- $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \Theta(n^3)$
- $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 = \Theta(n^4)$
- $\displaystyle\sum_{i=1}^{n} i^4 = \qquad = \Theta(n^5)$

$\vdots$

Proof.

- $$\sum_{i=1}^{n} i^k \le \sum_{i=1}^{n} n^k = n \cdot n^k = n^{k+1} = O(n^{k+1})$$

- $$\sum_{i=1}^{n} i^k \ge \sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^k$$

$$\ge \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \left\lceil \frac{n}{2} \right\rceil^k$$

$$= \left(n - \left\lceil \frac{n}{2} \right\rceil + 1\right) \cdot \left\lceil \frac{n}{2} \right\rceil^k$$

$$= \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) \left\lceil \frac{n}{2} \right\rceil^k$$

$$> \left(\frac{n}{2} - \cancel{1} + \cancel{1}\right) \left(\frac{n}{2}\right)^k \qquad \begin{cases} \lfloor x \rfloor > x - 1 \\ \lceil x \rceil \ge x \end{cases}$$

$$= \left(\frac{n}{2}\right)^{k+1}$$

$$= \left(\frac{1}{2}\right)^{k+1} \cdot n^{k+1} = \Omega(n^{k+1})$$

so $$\sum_{i=1}^{n} i^k = \Theta(n^{k+1}).$$ ∎

<u>Exercise</u>

let $f(i) = \Theta(i)$. Then

$$\sum_{i=1}^{n} f(i) = \Theta(n^2).$$

<u>Defn</u>

$o(g(n)) =$

$\{f(n) \mid \forall c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq f(n) < c\,g(n)\}$

Recall:

$O(g(n)) =$

$\{f(n) \mid \exists c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq f(n) \leq c\,g(n)\}$

<u>note</u>: $o(g(n)) \subseteq O(g(n))$.

<u>Lemma</u>

$f(n) = o(g(n))$ iff $\lim_{n\to\infty} \left(\frac{f(n)}{g(n)}\right) = 0$

<u>Proof</u>

$f(n) = o(g(n))$ iff

$\forall c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq f(n) < c\,g(n)$

iff

$\forall c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq \frac{f(n)}{g(n)} < c$

which is defn of limit stmt. $\blacksquare$

Ex. $\ln(n) = o(n)$

$$\lim_{n\to\infty}\left(\frac{\ln(n)}{n}\right) = \lim_{n\to\infty}\left(\frac{1/n}{1}\right) = 0 .$$

Ex. $n^k = o(e^n)$ for $k \geq 1$.

$$\lim_{n\to\infty}\left(\frac{n^k}{e^n}\right) = \lim_{n\to\infty}\left(\frac{k\cdot n^{k-1}}{e^n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{k(k-1)\,n^{k-2}}{e^n}\right)$$

$$\vdots$$

$$= 0$$

after $\lceil k \rceil$ applications of l'Hop.

<u>Exercise</u>

- $o(g(n)) \cap \Omega(g(n)) = \phi$

<u>Defn</u>

$\omega(g(n)) =$

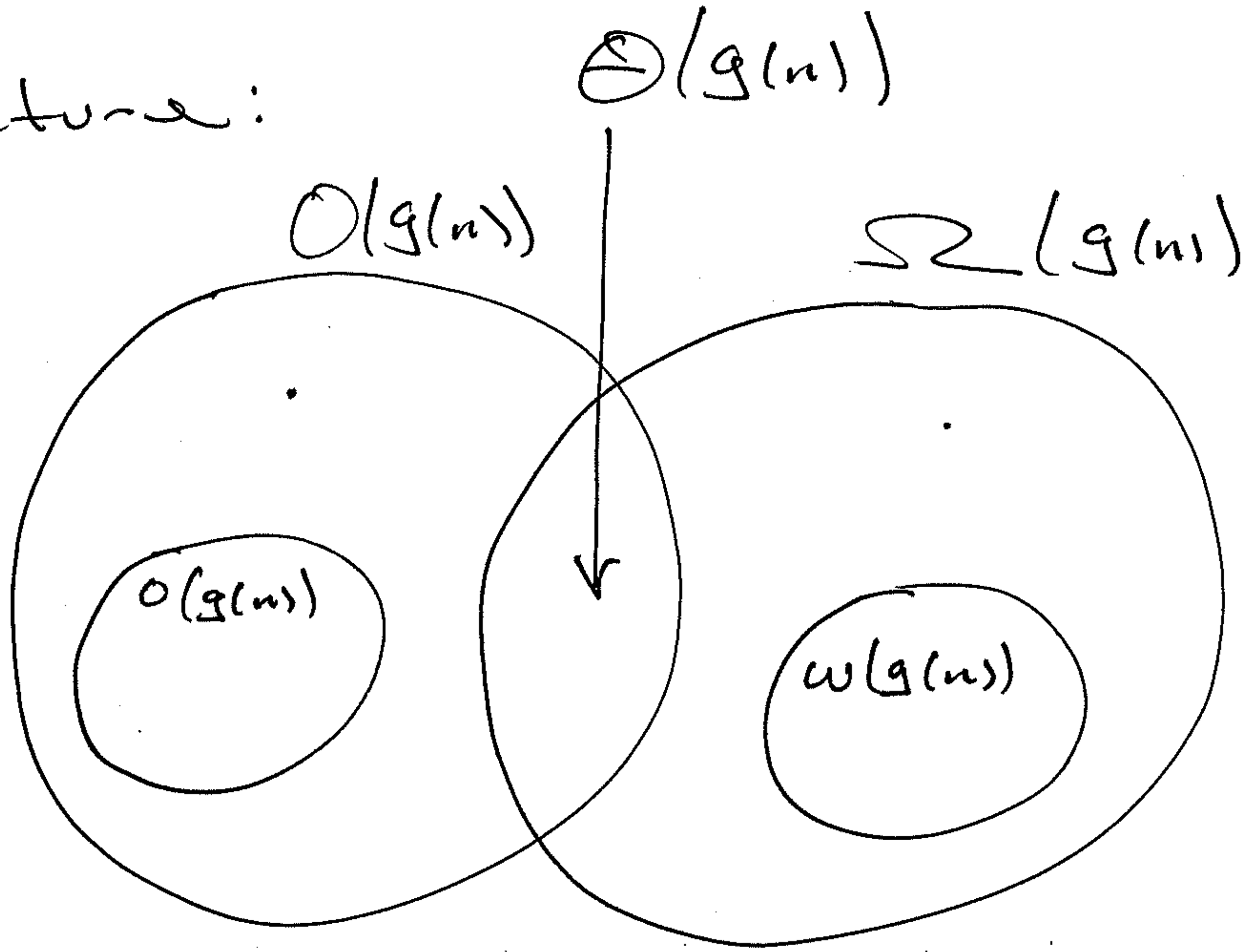
$\{f(n) \mid \forall c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq c g(n) < f(n)\}$

<u>Exercise</u>

- $f(n) = \omega(g(n))$ iff $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = \infty$

- $\omega(g(n)) \cap O(g(n)) = \phi$

Picture:



<u>Theorem</u>

If $\lim_{n \to \infty} \left(\frac{f(n)}{g(n)} \right) = L$ where $0 \le L < \infty$,

then $f(n) = O(g(n))$

<u>Remark</u>: Converse is <u>false</u>!

<u>Proof:</u>

The limit says

$$\forall \varepsilon > 0, \exists n_0 > 0, \forall n \geq n_0 : \left| \frac{f(n)}{g(n)} - L \right| < \varepsilon$$

Let $\varepsilon = 1$. Then

$$\exists n_0 > 0, \forall n \geq n_0 : \left| \frac{f(n)}{g(n)} - L \right| < 1$$

$$\therefore \quad " \qquad " \quad : -1 < \frac{f(n)}{g(n)} - L < 1$$

$$\therefore \quad " \qquad " \quad : \frac{f(n)}{g(n)} < (1 + L)$$

$$\therefore \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq (1+L)\, g(n)$$

Let $c = (1+L)$ in defn. of $f(n) = O(g(n))$.

$\blacksquare$

Thm

If $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = L$ where $0 < L \leq \infty$,

then $f(n) = \Omega(g(n))$.

Proof.

Since $0 < L \leq \infty$, we let $L' = \frac{1}{L}$,

then $0 \leq L' < \infty$, and

$$\lim_{n\to\infty}\left(\frac{g(n)}{f(n)}\right) = \frac{1}{L} = L' \text{ and } 0 \leq L' < \infty.$$

By Prev. Thm: $g(n) = O(f(n))$

By another Thm: $f(n) = \Omega(g(n))$.

$\blacksquare$

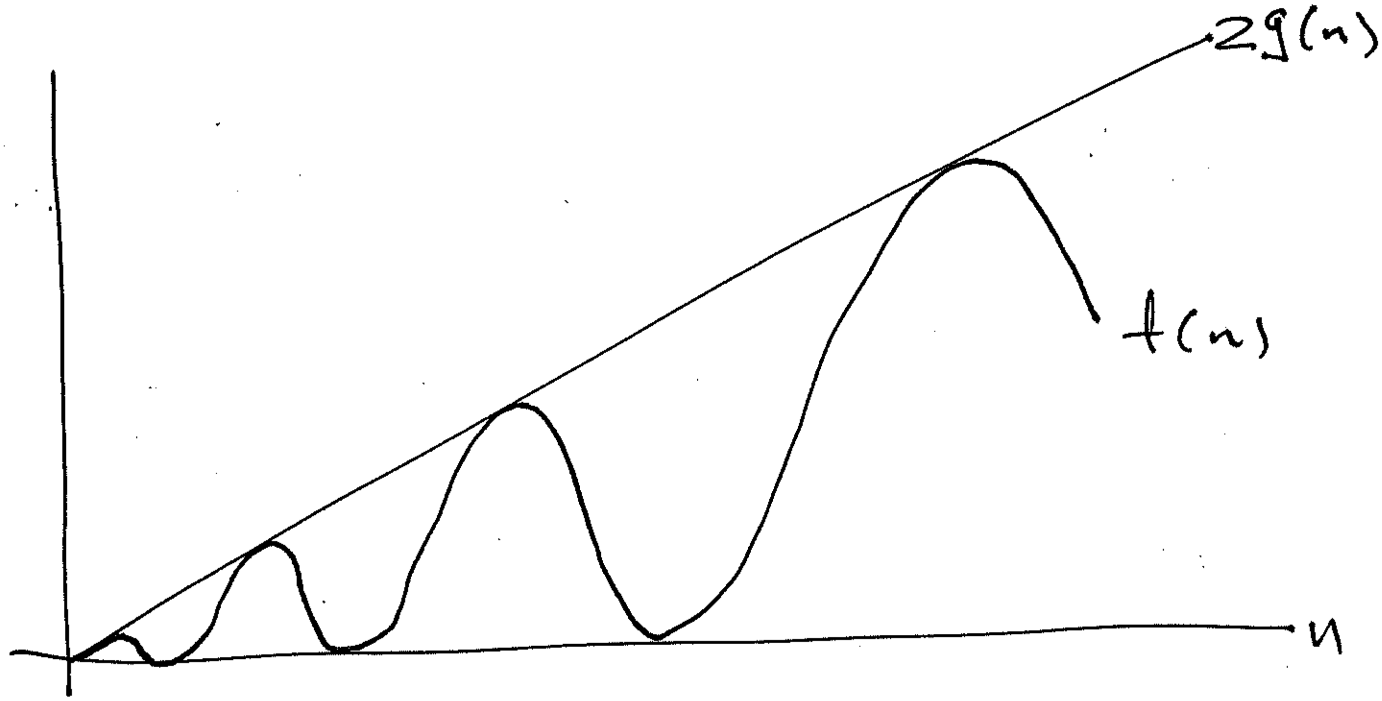
<u>Thm</u>

If $\lim_{n\to\infty} \left(\frac{f(n)}{g(n)}\right) = L$ where $0 < L < \infty$,

then $f(n) = \Theta(g(n))$.

<u>Rmk</u>. converse is false.

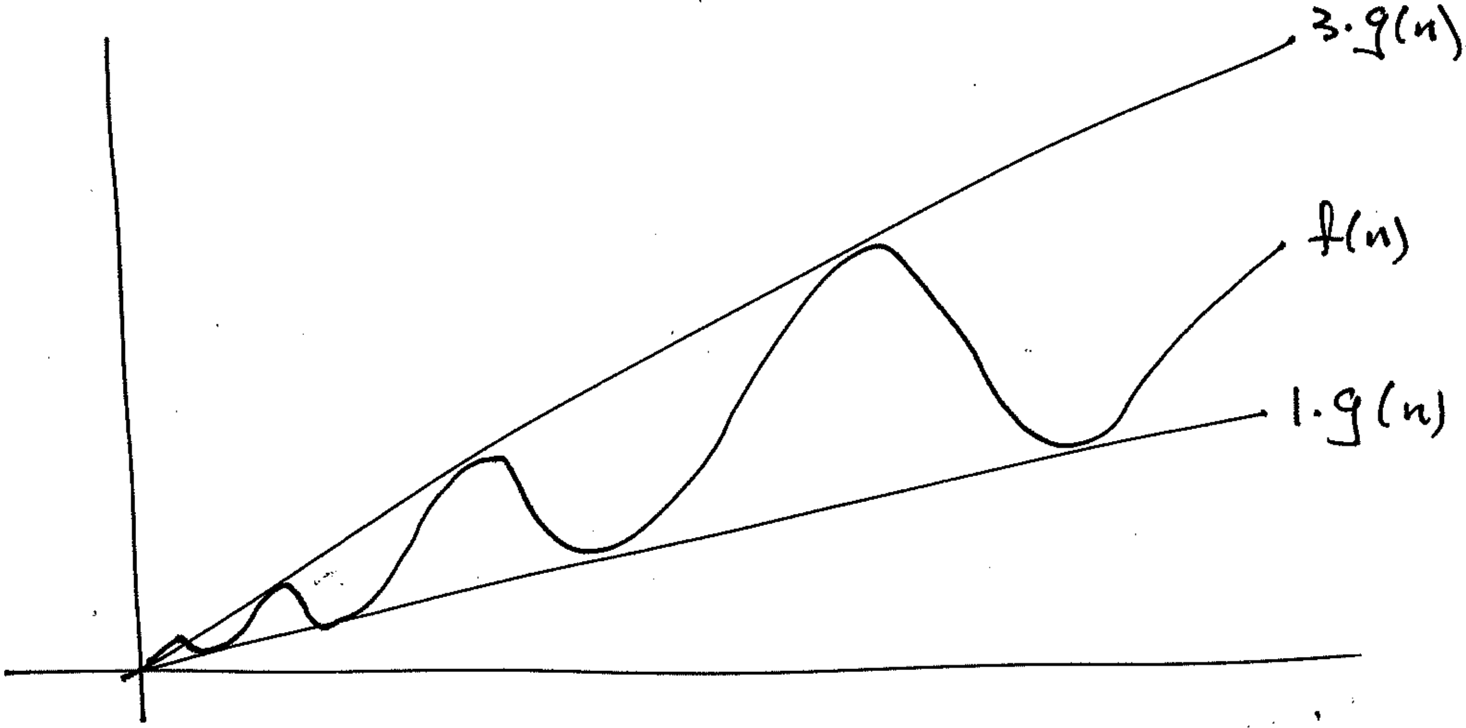
<u>Ex. A</u>. $g(n) = n$, $f(n) = (1+\sin(n))n$



so $f(n) = O(g(n))$ and $f(n) \neq \Omega(g(n))$

But $\frac{f(n)}{g(n)} = 1 + \sin(n)$, lim D.N.E.

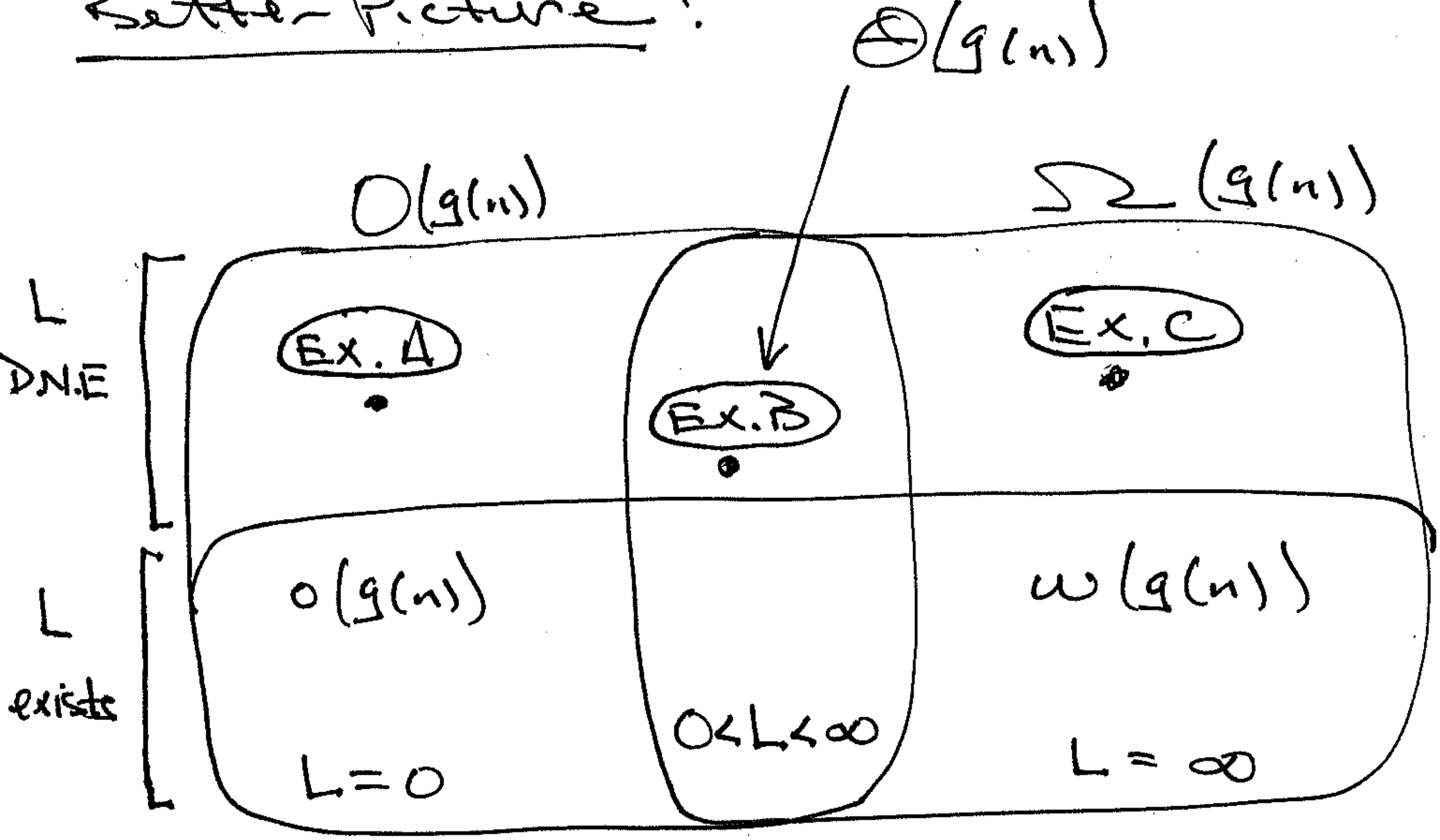
Ex. B : $g(n) = n$, $f(n) = (2 + \sin(n)) \cdot n$



$\therefore \quad f(n) = \Theta(g(n))$,

But $\frac{f(n)}{g(n)} = 2 + \sin(n)$, lim. D.N.E.

Better Picture:



Exercise find Example C, i.e.

fcns. $f(n)$, $g(n)$ s.t.

$$f(n) = \Omega(g(n))$$
$$f(n) \neq O(g(n))$$

But $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right)$ D.N.E.

<u>Recall</u>: if $a, b \in \mathbb{R}$, $b > 0$, then

$$(n+a)^b = \Theta(n^b)$$

<u>Proof</u>.

$$\lim_{n\to\infty} \frac{(n+a)^b}{n^b} = \lim_{n\to\infty} \left(\frac{n+a}{n}\right)^b$$

$$= \left(\lim_{n\to\infty} \frac{n+a}{n}\right)^b$$

$$= \left(\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)\right)^b$$

$$= 1^b = 1 \in (0, \infty)$$

$\therefore$ $(n+a)^b = \Theta(n^b)$. ∎

Ex

let $a > 1$ and $b > 1$. Then

$$a^{b^n} = \begin{cases} \omega\left(b^{a^n}\right) & \text{if } a < b \\ \Theta\left(b^{a^n}\right) & \text{if } a = b \\ o\left(b^{a^n}\right) & \text{if } a > b \end{cases}$$

$\exp_a(\exp_b(n))$

First do:

$$a^n = \begin{cases} o(b^n) & \text{if } a < b \ \checkmark \\ \Theta(b^n) & \text{if } a = b \ \checkmark \\ \omega(b^n) & \text{if } a > b \ \checkmark \end{cases}$$

Proof (of 2nd)

$$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n \xrightarrow[n\to\infty]{\text{as}} \begin{cases} 0 & \text{if } a<b \\ 1 & \text{if } a=b \\ \infty & \text{if } a>b \end{cases}$$