CSE 201 1-7-20

Asymptotic growth of functions:

<u>Defn.</u> Let $f(n)$ be a function

We say $f(n)$ is <u>asymptotically non-negative</u> (a.n.n.) iff

$$\exists n_0 > 0, \forall n \geq n_0 : \quad f(n) \geq 0$$

We say $f(n)$ is <u>asymptotically Positive</u> (a.p.) iff

$$\exists n_0 > 0, \forall n \geq n_0 : \quad f(n) > 0$$

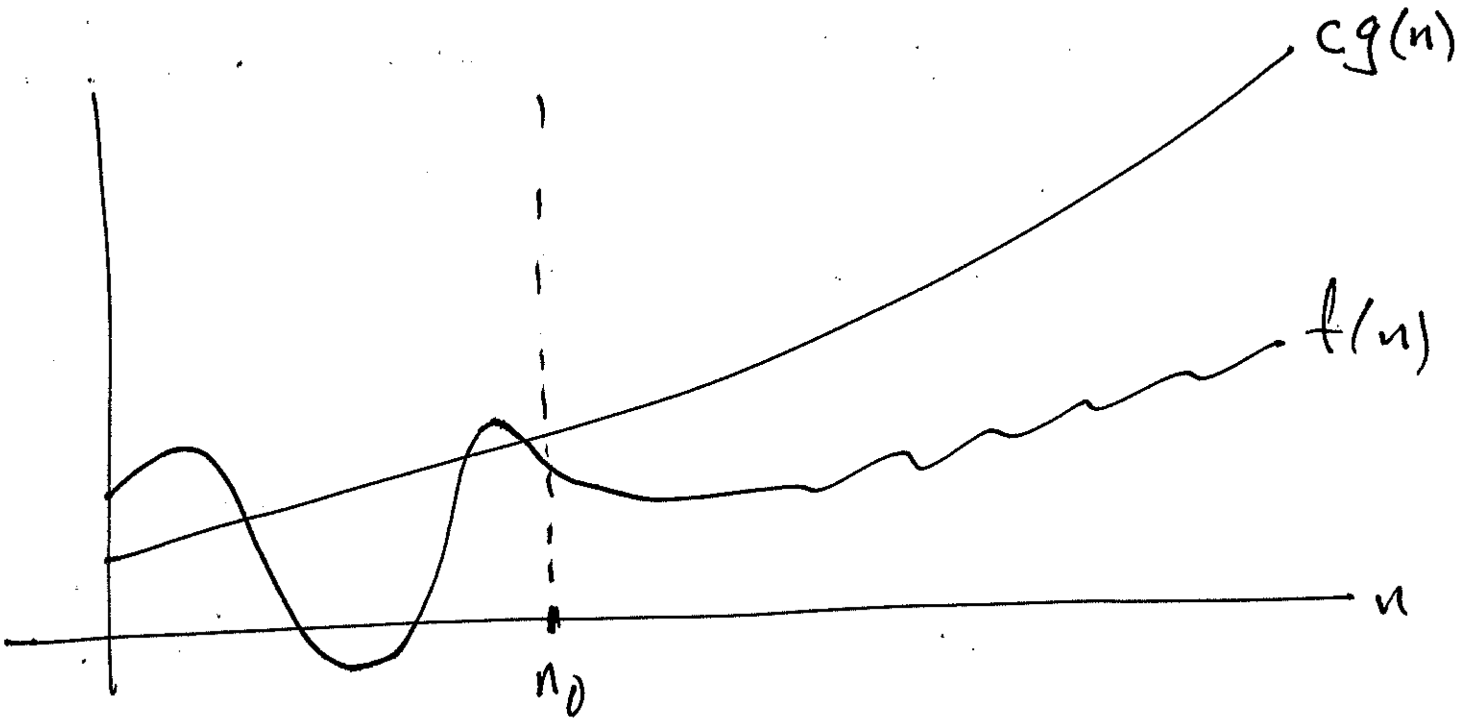
Defn

$O(g(n)) =$

$$\{ f(n) \mid \exists c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq f(n) \leq c\,g(n) \}$$

write $f(n) = O(g(n))$.

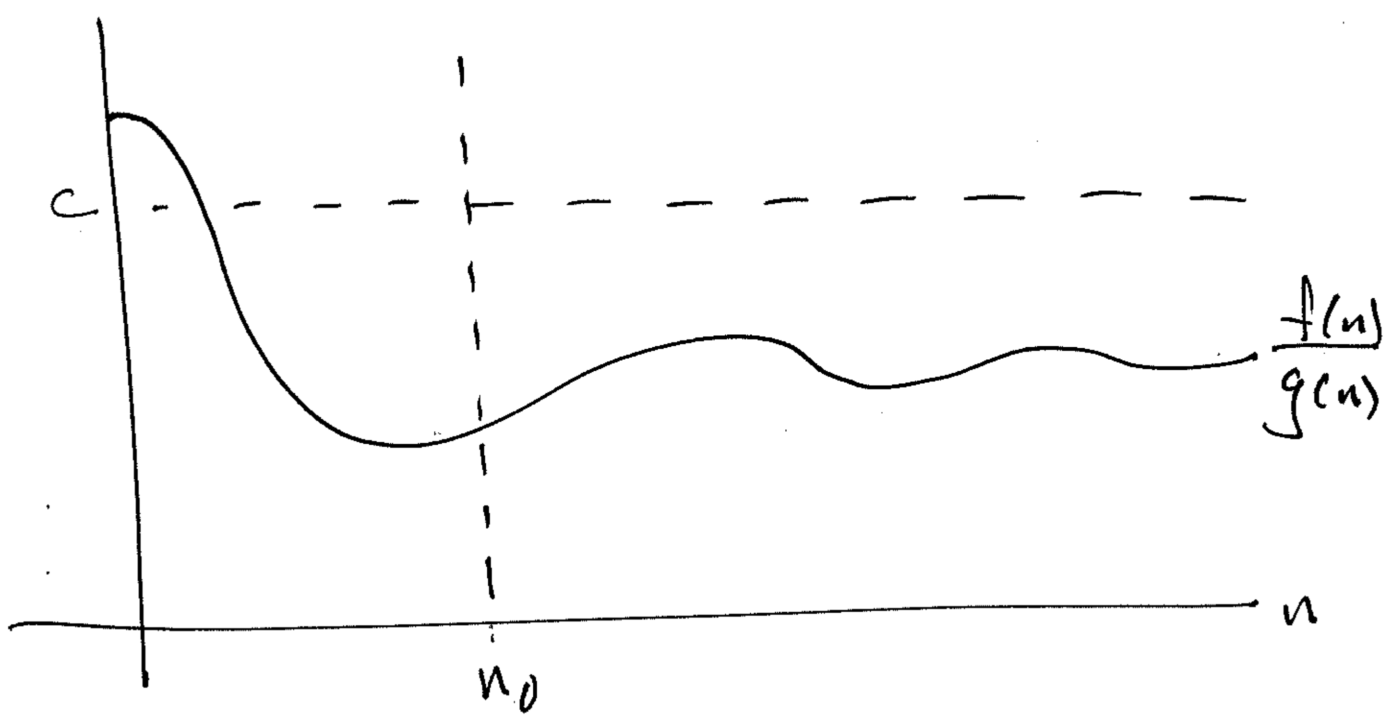
we say "$g(n)$ is an asymptotic upper bound for $f(n)$".



Blanket assumption: all fcns. under discussion are a.n.n.

<u>Note</u> if $g(n)$ is a.p., this is equivalent to

$$\exists c>0, \exists n_0>0, \forall n \geq n_0: \; 0 \leq \frac{f(n)}{g(n)} \leq c$$



Defn

$\Omega(g(n)) =$

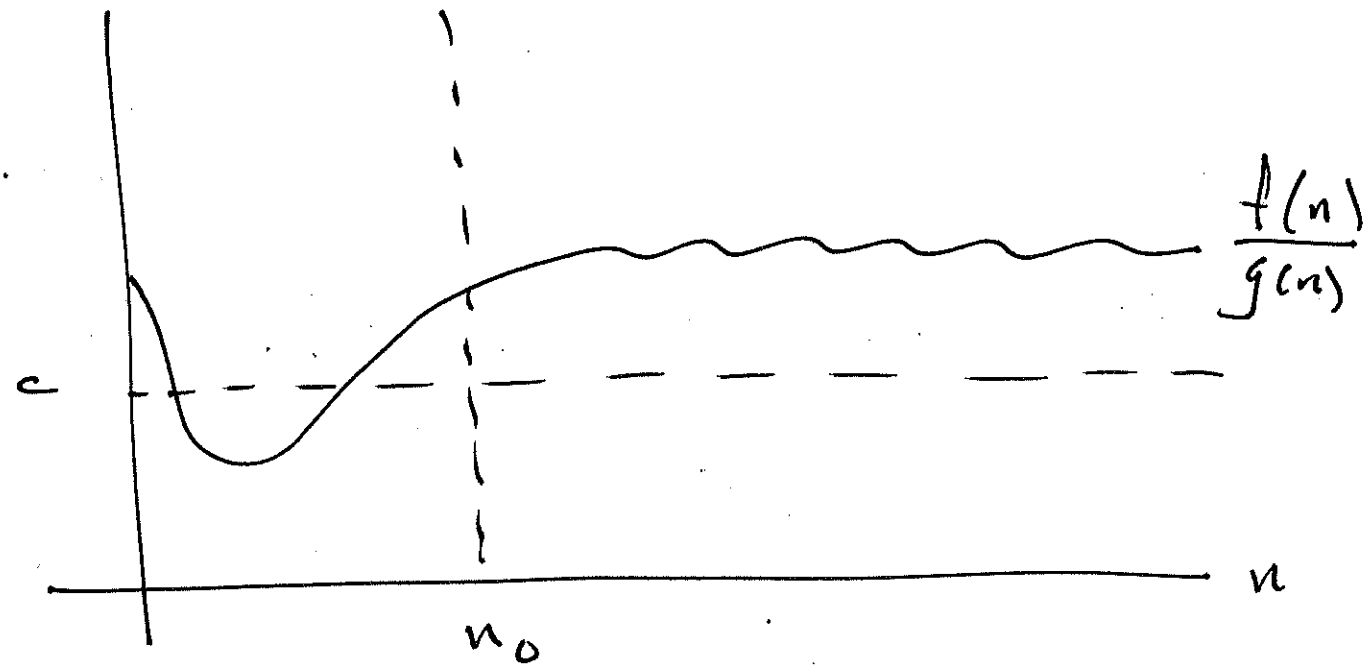
$\{f(n) \mid \exists c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq c\,g(n) \leq f(n)\}$

write $f(n) = \Omega(g(n))$

"asymptotic lower bound for $f(n)$"

note: if $g(n)$ is a.p., this is equiv. to

$$\exists c>0, \exists n_0>0, \forall n \geq n_0 : 0 \leq c \leq \frac{f(n)}{g(n)}$$



<u>Ex.</u> if $P(n), q(n)$ are polynomials

with

$$\deg(P(n)) \leq \deg(q(n))$$

then

$$P(n) = O(q(n))$$

and $$q(n) = \Omega(P(n))$$

Theorem

$f(n) = O(g(n)) \quad \underset{\longleftarrow}{\overset{\longrightarrow}{\text{iff}}} \quad g(n) = \Omega(f(n))$

Proof $(\Rightarrow)$

Assume $f(n) = O(g(n))$. Then

(1) $\exists c_1 > 0, \exists n_1 > 0, \forall n \geq n_1 : 0 \leq f(n) \leq c_1 g(n)$.

We must show $g(n) = \Omega(f(n))$, i.e.

(2) $\exists c_2 > 0, \exists n_2 > 0, \forall n \geq n_2 : 0 \leq c_2 f(n) \leq g(n)$.

Let $c_2 = \dfrac{1}{c_1}$, and $n_2 = n_1$.

Then $c_2 > 0$, $n_2 > 0$ and $(1) \Rightarrow (2)$.

Defn

$$\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$$

write $f(n) = \Theta(g(n))$

say "$g(n)$ is a tight asymptotic bound on $f(n)$"

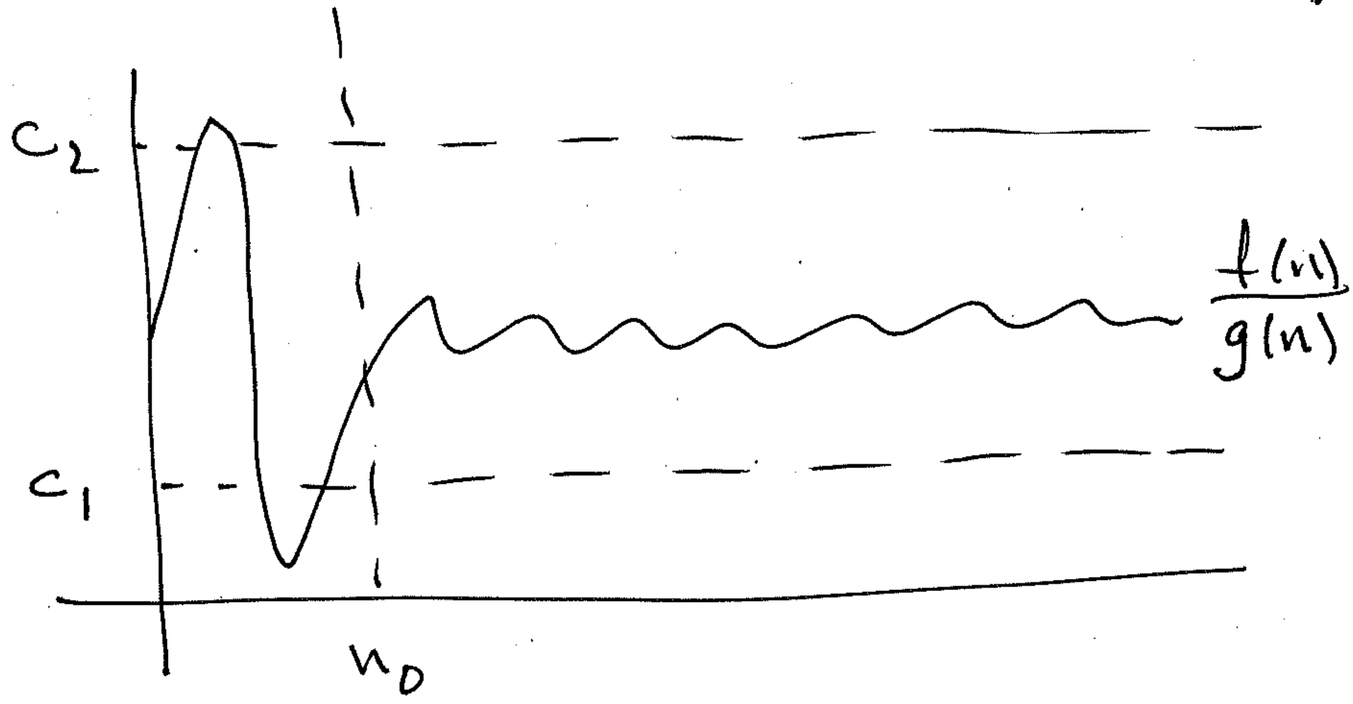
Equivalently: $f(n) = \Theta(g(n))$ iff

$$\exists c_1 > 0, \exists c_2 > 0, \exists n_0 > 0, \forall n \geq n_0:$$

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$$

If $g(n)$ is a.p. this is equiv. to

$$0 \leq c_1 \leq \frac{f(n)}{g(n)} \leq c_2$$



<u>Exercise</u> Prove

$$f(n) = \Theta(g(n)) \ \overset{\longrightarrow}{\underset{\longleftarrow}{\text{iff}}} \ g(n) = \Theta(f(n))$$

<u>Exercise</u> let $g(n)$ be any function, let $c > 0$. Then

$$c\, g(n) = O(g(n))$$
$$c\, g(n) = \Omega(g(n))$$
$$c\, g(n) = \Theta(g(n))$$

Analogy:

$f(n) = O(g(n)) \sim x \le y$

$f(n) = \Omega(g(n)) \sim x \ge y$

$f(n) = \Theta(g(n)) \sim x = y$

$f(n) = o(g(n)) \sim x < y$

$f(n) = \omega(g(n)) \sim y < x$

Ex. Prove $\sqrt{n+10} = \Theta(\sqrt{n})$

Proof

we must find positive c_1, c_2, n_0 s.t. for all $n \ge n_0$:

$$0 \le c_1\sqrt{n} \le \sqrt{n+10} \le c_2\sqrt{n}$$

Let $c_1 = 1$, $c_2 = \sqrt{2}$, $n_0 = 10$.

Then if $n \geq n_0 = 10$, we have

$$-10 \leq 0 \quad \text{and} \quad 10 \leq n$$

$$\therefore -10 \leq (1-1)n \quad \text{and} \quad 10 \leq (2-1)n$$

$$\therefore -10 \leq (1-c_1^2)n \quad \text{and} \quad 10 \leq (c_2^2-1)n$$

$$\therefore c_1^2 n \leq n+10 \quad \text{and} \quad n+10 \leq c_2^2 n$$

$$\therefore 0 \leq c_1^2 n \leq n+10 \leq c_2^2 n$$

$$\therefore 0 \leq c_1\sqrt{n} \leq \sqrt{n+10} \leq c_2\sqrt{n},$$

as required. ∎

Exercise

show $c_1 = \sqrt{\frac{1}{2}}$, $c_2 = \sqrt{\frac{3}{2}}$, $n_0 = 20$

Also work.

Exercise

let $a, b \in \mathbb{R}$ with $b > 0$. show that

$$(n+a)^b = \Theta(n^b)$$

Theorem

If $h(n) = O(g(n))$ and $f(n) \leq h(n)$ for all sufficiently large n,

then $f(n) = O(g(n))$.

Proof

Since $h(n) = O(g(n))$ we have

(1) $\exists c_1 > 0, \exists n_1 > 0, \forall n \geq n_1 : 0 \leq h(n) \leq c_1 g(n)$.

Since $f(n) \leq h(n)$ for suff. large n we have

(2) $\exists n_2 > 0, \forall n \geq n_2 : 0 \leq f(n) \leq h(n)$

we must show: $f(n) = O(g(n))$, i.e.

(3) $\exists\, c_3 > 0, \exists\, n_3 > 0, \forall n \geq n_3$:

$$0 \leq f(n) \leq c_3 g(n)$$

Let $c_3 = c_1$, $n_3 = \max(n_1, n_2)$.

Then $c_3 > 0$, $n_3 > 0$ and (1) and (2) imply (3). ∎

<u>Exercise</u> Show

if $h(n) = \Omega(g(n))$ and $f(n) \geq h(n)$ for all suff. large n, then

$$f(n) = \Omega(g(n)).$$