CSE 201 1-23-20

<u>Recall</u>

$$T(n) = \begin{cases} 1 & 1 \le n < 3 \\ 2T(\lfloor \frac{n}{3} \rfloor) + n & n \ge 3 \end{cases}$$

$\vdots$

$$T(n) = \sum_{i=0}^{k-1} 2^i \left(\frac{n}{3^i}\right) + 2^k T\left(\frac{n}{3^k}\right)$$

$$= n \cdot \sum_{i=0}^{k-1} \left(\frac{2}{3}\right)^i + 2^k T\left(\frac{n}{3^k}\right)$$

note: recursion ends when

$$1 \le \frac{n}{3^k} < 3$$

$\therefore \quad 3^k \leq n < 3^{k+1}$

$\therefore \quad k \leq \log_3(n) < k+1$

$\therefore \quad k = \lfloor \log_3(n) \rfloor$

so

$$T(n) = n \cdot \sum_{i=0}^{k-1} \left(\tfrac{2}{3}\right)^i + 2^{\log_3 n} \cdot 1$$

$$= n\left(\sum_{i=0}^{k-1} \left(\tfrac{2}{3}\right)^i\right) + n^{\log_3(2)}$$

$$= n\left(\frac{1-(2/3)^k}{1-(2/3)}\right) + n^{\log_3(2)}$$

$$= 3n\left(1-\left(\tfrac{2}{3}\right)^k\right) + n^{\log_3(2)}$$

$\therefore \; T(n) \leq 3n + n^{\log_3(2)} = O(n)$

Guess: $T(n) = O(n)$

<u>Exercise</u>: use substitution method to show $T(n) = O(n)$.

<u>Also</u> $T(n) = 2T(\lfloor \frac{n}{3} \rfloor) + n \geq n = \Omega(n)$

<u>Result</u>: $T(n) = \Theta(n)$.

Iteration

Ex. $T(n) = \begin{cases} 1 & 1 \le n < 3 \\ 2T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n & n \ge 3. \end{cases}$

$$
\begin{aligned}
T(n) &= n + 2\,T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) \\
&= n + 2\left(\left\lfloor \frac{n}{3} \right\rfloor + 2T\left(\left\lfloor \frac{\lfloor n/3 \rfloor}{3} \right\rfloor\right)\right) \\
&= n + 2\left\lfloor n/3 \right\rfloor + 2^2 T\left(\left\lfloor \frac{n}{3^2} \right\rfloor\right) \\
&= n + 2\left\lfloor \frac{n}{3} \right\rfloor + 2^2\left(\left\lfloor \frac{n}{3^2} \right\rfloor + 2T\left(\left\lfloor \frac{\lfloor n/3^2 \rfloor}{3} \right\rfloor\right)\right) \\
&= n + 2\left\lfloor \frac{n}{3} \right\rfloor + 2^2\left\lfloor \frac{n}{3^2} \right\rfloor + 2^3 T\left(\left\lfloor \frac{n}{3^3} \right\rfloor\right) \\
&\;\vdots \\
&= \sum_{i=0}^{k-1} 2^i \left\lfloor \frac{n}{3^i} \right\rfloor + 2^k \cdot T\left(\left\lfloor \frac{n}{3^k} \right\rfloor\right)
\end{aligned}
$$

Stop when: $1 \le \left\lfloor \frac{n}{3^k} \right\rfloor < 3$

$$\Rightarrow 1 \le \frac{n}{3^k} < 3 \Rightarrow \dots \Rightarrow k = \lfloor \log_3(n) \rfloor \le \log_3 n$$

for this k:

$$T(n) = \sum_{i=0}^{k-1} 2^i \left\lfloor \frac{n}{3^i} \right\rfloor + 2^k \cdot 1$$

$$\le \sum_{i=0}^{k-1} 2^i \left(\frac{n}{3^i} \right) + 2^{\log_3 n}$$

$$\le n \left(\sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^i \right) + n^{\log_3(2)}$$

$$= n \left(\frac{1}{1 - 2/3} \right) + n^{\log_3(2)}$$

$$= 3n + n^{\log_3(2)} = O(n)$$

$$\therefore T(n) = O(n)$$

Also $T(n) = 2T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n \geq n = \Omega(n)$

$\therefore T(n) = \Omega(n)$

$\therefore T(n) = \Theta(n)$. $\left\{ \text{asymptotic solution} \right.$.

<u>Ex</u>

$$T(n) = \begin{cases} 5 & 0 \leq n < 2 \\ T(n-2) + n & n \geq 2 \end{cases}$$

$$\begin{aligned} T(n) &= n + T(n-2) \\ &= n + (n-2) + T(n-4) \\ &= n + (n-2) + (n-4) + T(n-6) \end{aligned}$$

$\vdots$

$$= \sum_{i=0}^{k-1} (n-2i) + T(n-2k)$$

$$= \sum_{i=0}^{k-1} n - 2\sum_{i=\cancel{0}\,1}^{k-1} i + T(n-2k)$$

$$= kn - \cancel{2}\left(\frac{k(k-1)}{\cancel{2}}\right) + T(n-2k)$$

stop when $0 \le n-2k < 2$

$$2k \le n < 2k+2$$

$$k \le \frac{n}{2} < k+1$$

$$\therefore \quad k = \left\lfloor \frac{n}{2} \right\rfloor$$

Exact solution

$$\therefore T(n) = \left\lfloor \frac{n}{2} \right\rfloor n - \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) + 5$$

$$= \Theta(n^2)$$

asymptotic solution

$$T(n) \sim \frac{1}{2}n^2 - \frac{1}{4}n^2 + \frac{1}{2}n + 5$$

$$= \frac{1}{4}n^2 + \frac{1}{2}n + 5$$

Master Method

Theorem let $a \geq 1$, $b > 1$, $f(n)$ a.p. and suppose $T(n)$ is defined by

$$T(n) = aT\left(\frac{n}{b}\right) + f(n).$$

Then we have 3 cases:

(1) if $f(n) = O(n^{\log_b a - \varepsilon})$ for some $\varepsilon > 0$

then $\boxed{T(n) = \Theta(n^{\log_b a})}$.

(2) if $f(n) = \Theta(n^{\log_b a})$, then

$$\boxed{T(n) = \Theta(f(n) \cdot \log n) = \Theta(n^{\log_b a} \cdot \log n)}$$

(3) if $f(n) = \Omega(n^{\log_b a + \varepsilon})$, for some $\varepsilon > 0$, and if

regularity condition

$a f(n/b) \le c f(n)$ for some c in $0 < c < 1$ and all suff. large n, then

$$\boxed{T(n) = \Theta(f(n))}$$

Remark:

we compare: $f(n)$ to $n^{\log_b a}$

Ex. $T(n) = T(\lfloor \frac{n}{2} \rfloor) + 2T(\lceil \frac{n}{2} \rceil) + \log(n!)$

Simplify:

$$T(n) = 3T\left(\frac{n}{2}\right) + n\log n$$

compare: $n\log n$ to $n^{\log_2 3}$

Recall $\frac{\log n}{n^{\varepsilon}} \longrightarrow 0 \quad \forall \varepsilon > 0$

let $\varepsilon = \frac{1}{2}(\log_2 3 - 1) > 0$

Then $2\varepsilon = \log_2 3 - 1$

$$1 + \varepsilon = \log_2 3 - \varepsilon$$

so

$$\frac{n \log n}{n^{\log_2 3 - \varepsilon}} = \frac{n \log n}{n^{1+\varepsilon}} = \frac{\log n}{n^{\varepsilon}} \to 0 \quad \text{as } n \to \infty$$

$\therefore$ $n \log n = o(n^{\log_2 3 - \varepsilon}) \subseteq O(n^{\log_2 3 - \varepsilon})$

so by case 1:

$$T(n) = \Theta(n^{\log_2 3}).$$

<u>Exercise</u>

Prove that if $f(n) = \Theta(n^d)$ with $d \geq 0$, and $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some $\varepsilon > 0$, then the regularity condition must hold, i.e. $\exists c$ s.t. $0 < c < 1$ and

$$a f(n/b) \leq c f(n)$$

for all suff. large n.

Exercise

find an example $(a, b, f(n))$ where $\exists \varepsilon > 0$ s.t.

$$f(n) = \Omega\left(n^{\log_b a + \varepsilon}\right)$$

but regularity condition fails.