cse 201 1-16-20

Ex. define

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ T(\lfloor \frac{n}{2} \rfloor) + 1 & \text{if } n \geq 2 \end{cases}$$

show: $\boxed{T(n) \leq \lg(n)}$ ← $P(n)$ for all $n \geq 1$.

(hence $T(n) = O(\log n)$.)

Proof.

I. $P(1)$ says

$T(1) \leq \lg(1)$ ✓

i.e. $0 \leq 0$ ✓

II d. $\forall n > 1 : \underbrace{P(1) \wedge \ldots \wedge P(n-1)}_{\forall k:\, 1 \le k < n:\, P(k)} \rightarrow P(n)$

- Let $n > 1$ be chosen arbitrarily.
- Assume for any k in the range $1 \le k < n$ that $T(k) \le \lg(k)$.
- We must show $T(n) \le \lg(n)$.

Then

$$T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1$$ by the recurrence for $T(n)$

$$\le \lg\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1$$ • By the ind. hyp. with $k = \lfloor \frac{n}{2} \rfloor$

$$\le \lg\left(\frac{n}{2}\right) + 1$$ since $\lfloor x \rfloor \le x$ and since $\lg(\cdot)$ is increasing

$= \lg(n) - \cancel{1} + \cancel{1}$

$= \lg(n)$

Hence $T(n) \le \lg(n)$ for all $n \ge 1$ by 2^{nd} P.M.I. $\blacksquare$

Multiple Base Cases:

I. show $P(1), P(2), \dots, P(n_0)$

IId. $\forall n > n_0 : P(1) \wedge \dots \wedge P(n-1) \rightarrow P(n)$

Conclusion: $\forall n \ge 1 : P(n)$.

Ex. define

$$T(n) = \begin{cases} 1 & \text{if } n=1, n=2 \\ 4T(\lfloor \frac{n}{3} \rfloor) + n & \text{if } 3 \le n \end{cases}$$

show $\forall n \ge 1 : \boxed{T(n) \le n^2} \leftarrow P(n)$

(hence $T(n) = O(n^2)$.)

Proof..

I. $P(1)$ says: $T(1) \le 1^2$

$1 \le 1$ ✓

$P(2)$ says: $T(2) \le 2^2$

$1 \le 4$ ✓

II d. $\forall n > 2 : P(1) \wedge \ldots \wedge P(n-1) \rightarrow P(n)$

- Let $n > 2$ be arbitrary.
- Assume for all k in range $1 \le k < n$ that $T(k) \le k^2$.
- We must show that: $T(n) \le n^2$

Then

$$T(n) = 4T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n \quad \text{by def'n of } T(n).$$

$$\le 4 \cdot \left(\left\lfloor \frac{n}{3} \right\rfloor\right)^2 + n$$

by ind. hyp. using $k = \left\lfloor \frac{n}{3} \right\rfloor$

<u>note</u>: $1 \le \left\lfloor \frac{n}{3} \right\rfloor < n$

$$\le 4\left(\frac{n}{3}\right)^2 + n$$

Since $\lfloor x \rfloor \le x$ and $(\ )^2$ is increasing.

$$\le n^2$$

note $4\left(\frac{n}{3}\right)^2 + n \le n^2$

pf. $n > 2 \Rightarrow \frac{9}{5} \le n$

$$\Rightarrow n \le \frac{5}{9} n^2$$

$$\Rightarrow \frac{4}{9} n^2 + n \le n^2$$

$$\Rightarrow 4\left(\frac{n}{3}\right)^2 + n \le n^2 \checkmark$$

Hence $T(n) \le n^2$ for all $n \ge 1$

By 2^{nd} PMI. ∎

why 2 base cases?

n	1	2	3	4	5	6	7	8	9	10
$\lfloor \frac{n}{3} \rfloor$	0	0	1	1	1	2	2	2		

note: $P(n)$ depends on

$$P\left(\left\lfloor \frac{n}{3^{\lfloor \log_3 n \rfloor}} \right\rfloor\right)$$

Induction Fallacies

Example C:

All Horses are of the same color.

Prove

$P(n)$

$\forall n \geq 1$: $\boxed{\text{If } S \text{ is a set of } n \text{ horses, then all horses in } S \text{ are of same color.}}$

I. $P(1)$ says: "if S is a set of 1 horse, then all horses in S have same color".

IIa. $\forall n > 1 : P(n) \Rightarrow P(n+1)$

- let $n > 1$ be arbitrary, hence $n \geq 2$.
- Assume that if S is a set of n horses, all of them have same color.

- We must show: If S is a set of $n+1$ horses, then all of them have same color.

Let S be a set of $n+1$ horses, say

$$S = \{h_1, h_2, h_3, \ldots, h_{n+1}\}.$$

consider the sets

$$S' = S - \{h_1\}$$

$$S'' = S - \{h_2\}$$

~~the~~ each contain n horses, so

- by the ind. hyp., all in S' have same color, and all in S'' have same color.

<u>note</u> $h_3 \in S' \cap S''$

observe $n \geq 2 \Rightarrow n+1 \geq 3$ so h_3 exists. Since h_3 has only one color, the horses in S' have same color as horses in S''. Therefore, all horses in $S' \cup S'' = S$ have the same color. $\blacksquare$

Justification of Induction

Theorem 1: 1^{st} PMI.

For any propositional function $P(n)$, the following sentence is true

$$\left[P(1) \wedge \left(\forall n>1: P(n-1) \rightarrow P(n)\right)\right] \rightarrow \left(\forall n: P(n)\right)$$

Theorem 2: 2^{nd} PMI

for any Prop. fcn $P(n)$, the following sentence is true

$$\left[P(1) \wedge \left(\forall n>1: \left(\forall k<n: P(k)\right) \rightarrow P(n)\right)\right] \rightarrow \left(\forall n: P(n)\right)$$

Theorem 3 (well ordering Prop. of $\mathbb{Z}^+$: WOP)

Any non-empty subset of $\mathbb{Z}^+$ contains a least element.

Proof that WOP $\Rightarrow$ 1st PMI

Assume both

(1) $P(1)$

and

(2) $\forall n > 1 : P(n-1) \Rightarrow P(n)$

are true. Let

$$S = \{ n \in \mathbb{Z}^+ \mid P(n) \text{ is false} \}.$$

It is sufficient to show $S = \phi$, since then we have

$$\forall n : P(n)$$

is true.

Assume, to get a contradiction, that $S \neq \phi$. By the W.O.P. S contains a least element, call it m. So

- $m \in S$

and

- if $k < m$, then $k \notin S$

Since $m \in S$, $P(m)$ is false.

But $P(1)$ is true, so $m \neq 1$.

Hence $m > 1$, and $m-1 \geq 1$.

so $P(m-1)$ is defined.

Now $m-1 < m$, so $m-1 \notin S$,

therefore $\boxed{P(m-1)}$ is true.

By (2) and Univ. Inst. we have

that $\boxed{P(m-1) \rightarrow P(m)}$ is true.

So By Modus Ponens $P(m)$ is

true. It follows that

$m \notin S$. This contradicts

That $m \in S$. This ※ shows our assumption was false.

Hence $S = \emptyset$ and

$$\forall n : P(n)$$

is true. ∎