CSE 201 1-14-20

Ex. 9c let $a, b \in \mathbb{R}$, $a > 0$, $b > 0$.

$$a^n = \begin{cases} o(b^n) & \text{if } a < b \checkmark \\ \Theta(b^n) & \text{if } a = b \checkmark \\ \omega(b^n) & \text{if } a > b \checkmark \end{cases}$$

Proof

$$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n \xrightarrow[\text{as } n \to \infty]{} \begin{cases} 0 & \text{if } a < b \\ 1 & \text{if } a = b \\ \infty & \text{if } a > b \end{cases}$$

Ex. 7 (P.9)

$$a^{b^n} = \begin{cases} \omega\left(b^{a^n}\right) & a < b \checkmark \\ \Theta\left(b^{a^n}\right) & a = b \checkmark \\ o\left(b^{a^n}\right) & a > b \checkmark \end{cases}$$

Proof:

$$\lim_{n \to} \left(\frac{a^{b^n}}{b^{a^n}} \right) = \begin{cases} \infty & a < b \\ 1 & a = b \\ 0 & a > b \end{cases}$$

$$\ln\left(\frac{a^{b^n}}{b^{a^n}} \right) = \ln\left(a^{b^n}\right) - \ln\left(b^{a^n}\right)$$

$$= b^n \cdot \ln(a) - a^n \cdot \ln(b) \to \begin{cases} +\infty & a < b \\ 0 & a = b \\ -\infty & a > b \end{cases}$$

Some Common Functions (3.2)

log functions: let $a, b \in \mathbb{R}$, $a > 1$, $b > 1$.

$$x = a^{\log_a(x)} = \left(b^{\log_b(a)}\right)^{\log_a(x)}$$

$$= b^{\log_b(a) \cdot \log_a(x)}$$

$$\therefore \quad \log_b(x) = \underbrace{\log_b(a)}_{\text{const.}} \cdot \log_a(x)$$

$$\log_b(n) = \text{const.} \log_a(n)$$

$$\therefore \quad \log_b(n) = \Theta(\log_a(n))$$

Stirling's Formula

If $n \in \mathbb{Z}^+$, then

$$\boxed{n! = \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)}$$

Corollary

(a) $n! = \omega(n^n)$ ✓

(b) $n! = \omega(b^n)$ for any $b > 0$.

(c) $\log(n!) = \Theta(n \log n)$ ✓

Proof of (a)

$$\frac{n!}{n^n} = \frac{\sqrt{2\pi n}\cdot \cancel{n^n}/e^n \cdot \left(1+\Theta\left(\frac{1}{n}\right)\right)}{\cancel{n^n}}$$

$$= \sqrt{2\pi}\cdot \frac{n^{1/2}}{e^n}\cdot\left(1+\Theta\left(\frac{1}{n}\right)\right) \xrightarrow[n\to\infty]{\text{as}} 0$$

Proof of (e)

$$\log(n!) = \log\sqrt{2\pi} + \log n^{1/2} + \log n^n - \log e^n + \log\left(1+\Theta\left(\frac{1}{n}\right)\right)$$

$$= \log\sqrt{2\pi} + \frac{1}{2}\log n + n\log n - n\log e + \log\left(1+\Theta\left(\frac{1}{n}\right)\right)$$

$$\frac{\log(n!)}{n\log n} = \underbrace{\frac{\log\sqrt{2\pi}}{n\log n}}_{\downarrow\, 0} + \underbrace{\frac{1}{2n}}_{\downarrow\, 0} + \underbrace{1}_{\downarrow\, 1} - \underbrace{\frac{\log e}{\log n}}_{\downarrow\, 0} + \underbrace{\frac{\log\left(1+\Theta\left(\frac{1}{n}\right)\right)}{n\log n}}_{\downarrow\, 0} \longrightarrow 1$$

Ex. $\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$

note: $\binom{m}{k} \overset{\text{def.}}{=}$ # of k-subsets of an m-set. $(0 \le k \le m)$

$m = 3$: $S = \{1, 2, 3\}$

$k = 0$: ϕ $\qquad \binom{3}{0} = 1$

$k = 1$: $\{1\}, \{2\}, \{3\}$ $\qquad \binom{3}{1} = 3$

$k = 2$: $\{1, 2\}, \{1, 3\}, \{2, 3\}$ $\qquad \binom{3}{2} = 3$

$k = 3$: $\{1, 2, 3\}$ $\qquad \binom{3}{3} = 1$

Pascals Δ :

n	
0	1
1	1 1
2	1 2 1
$\boxed{3}$	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
$\vdots$	

$$\vdots$$

$$\binom{2n}{n}$$

__Thm__ $\binom{m}{k} = \dfrac{m!}{k!\,(m-k)!}$ $\Big\}$ See Discrete math text.

__Proof of $\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$:__

$$\binom{2n}{n} = \frac{(2n)!}{n!\,(2n-n)!} = \frac{(2n)!}{(n!)^2}$$

$$= \frac{\sqrt{2\pi\cdot 2n}\cdot\left(\frac{2n}{e}\right)^{2n}\cdot\left(1+\Theta\left(\frac{1}{2n}\right)\right)}{\left(\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^{n}\cdot\left(1+\Theta\left(\frac{1}{n}\right)\right)\right)^2}$$

$$= \frac{\cancel{2}\sqrt{\pi}\cdot\sqrt{n}}{\cancel{2}\cdot\pi\cdot n}\cdot\frac{2^{2n}\cdot\cancel{n^{2n}/e^{2n}}}{\cancel{n^{2n}/e^{2n}}}\cdot\frac{\left(1+\Theta\left(\frac{1}{2n}\right)\right)}{\left(1+\Theta\left(\frac{1}{n}\right)\right)^2}$$

$$= \frac{1}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{n}}\cdot 4^n\cdot\Big[\qquad\qquad\Big]$$

$$\therefore \quad \frac{\binom{2n}{n}}{\frac{4^n}{\sqrt{n}}} = \frac{1}{\sqrt{\pi}} \cdot \left[\qquad \right] \xrightarrow[\text{as } n \to \infty]{} \frac{1}{\sqrt{\pi}} \in (0, \infty)$$

$$\therefore \quad \binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right). \qquad \blacksquare$$

Exercise: show

$$\binom{3n}{n} = \Theta(\text{something})$$

Induction Proofs

Let $P(n)$ be a propositional function of $n \in \mathbb{N}$ (or some subset of $\mathbb{N}$).

$$\mathbb{N} = \{0, 1, 2, \ldots\ldots\}$$

$$\mathbb{Z}^{+} = \{1, 2, 3, \ldots\}$$

$$P : \mathbb{N} \longrightarrow \{\text{true}, \text{false}\}$$

<u>Goal</u>: Prove statements of the form

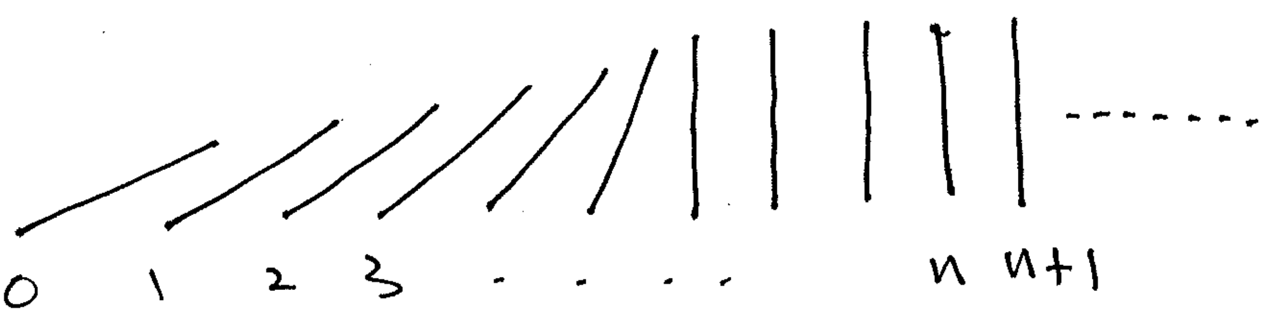
$$\forall n : P(n)$$

I. Prove Base case: $P(0)$ is true

II. Induction step: $\forall n \geq 0 : P(n) \rightarrow P(n+1)$

conclude $\forall n \; P(n)$.

<u>Domino analogy</u>



$P(n) =$ "the n^{th} domino falls."

I. $P(0) =$ "the 0^{th} domino falls"

II. $\forall n \geq 0 : P(n) \Rightarrow P(n+1)$

= "for any $n \in \mathbb{N}$, if the n^{th} domino falls, so does the $(n+1)^{st}$"

= "if any domino falls, so does the next".

Conclude: $\forall n : P(n)$

= "all dominos fall"

"Infinite" Proof

1.) $P(0)$ by I

2.) $P(0) \to P(1)$ by II

3.) $P(1)$ by Modus Ponens

4.) $P(1) \to P(2)$ by II

5.) $P(2)$ by M.P.

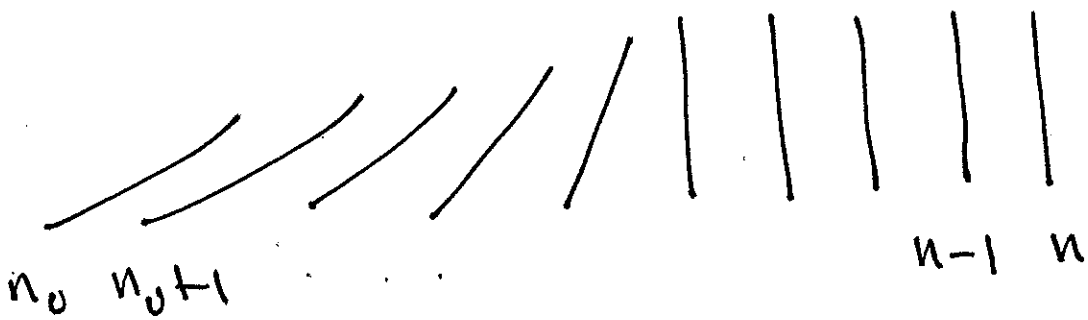
$\vdots$

Variation $P: \{n_0, n_0+1, n_0+2, \ldots\}$

I. $P(n_0)$

IIa. $\forall n \geq n_0 : \underbrace{P(n)}_{\text{induction hypothesis}} \to P(n+1)$

II b. $\forall n > n_0 : \underbrace{P(n-1)}_{\text{ind. hyp.}} \rightarrow P(n)$

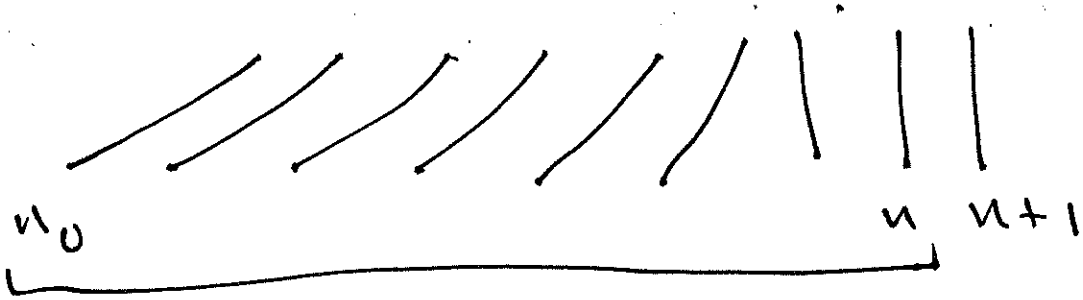


II a & II b are called <u>Weak Induction</u>

or <u>1st Principle of mathematical induction</u>

(1st PMI).

II c. $\forall n \geq n_0 : \overbrace{P(n_0) \wedge P(n_0+1) \wedge \ldots \wedge P(n)}^{\text{ind. hyp.}} \rightarrow P(n+1)$



or

$$\forall n \geq n_0 : \left(\forall k : n_0 \leq k \leq n : P(k)\right) \rightarrow P(n+1)$$

IId. $\forall n > n_0 : \overbrace{P(n_0) \wedge P(n_0+1) \wedge \dots \wedge P(n-1)}^{\text{ind. hyp.}} \longrightarrow P(n)$

IIc & IId are <u>strong induction</u>

or <u>2nd P.M.I.</u>