CSE 16
Winter 2022
Quiz 5 **Solutions**

1. (10 Points) Let $A = \{1, 2, 3, 4, 5, 6, 7\}$ and define a relation $R \subseteq A \times A$ by

$$R = \{(1,1), (1,2), (1,3), (2,3), (7,1), (7,2), (7,3), (4,5), (4,6), (5,6), (5,5)\}$$

Determine (**yes** or **no**) whether R has the following properties (no justification is required).

Reflexive? **no**
Symmetric? **no**
Transitive? **yes**
Anti-symmetric? **yes**

2. (20 Points) Use induction to prove $\forall n \geq 1$: $3 \mid (n^3 + 2n)$.

Proof: (Using weak induction form IIa.)

I. If $n = 1$, we have $n^3 + 2n = 1^3 + 2 \cdot 1 = 1 + 2 = 3$, which is divisible by 3, establishing the base case.

II. Let $n \geq 1$ be chosen arbitrarily. Assume, for this n, that $3 \mid (n^3 + 2n)$, whence $n^3 + 2n = 3k$ for some $k \in \mathbb{Z}$. We must show that $3 \mid \big((n+1)^3 + 2(n+1)\big)$. Observe that

$$\begin{aligned}(n+1)^3 + 2(n+1) &= (n^3 + 3n^2 + 3n + 1) + (2n + 2) \\ &= (n^3 + 2n) + (3n^2 + 3n + 3) \\ &= 3k + 3(n^2 + n + 1) \qquad \text{(by the induction hypothesis)} \\ &= 3(k + n^2 + n + 1),\end{aligned}$$

showing that $3 \mid \big((n+1)^3 + 2(n+1)\big)$, as required. ∎

Alternate Proof: (Using weak induction form IIb.)

I. Same as above.

II. Let $n > 1$ be chosen arbitrarily. Assume, for this n, that $3 \mid \big((n-1)^3 + 2(n-1)\big)$, and therefore $(n-1)^3 + 2(n-1) = 3k$ for some $k \in \mathbb{Z}$. We must show that $3 \mid (n^3 + 2n)$. Observe that

$$\begin{aligned}n^3 + 2n &= (n^3 - 3n^2 + 3n - 1) + (2n - 2) + (3n^2 - 3n + 1 + 2) \\ &= \big((n-1)^3 + 2(n-1)\big) + (3n^2 - 3n + 3) \\ &= 3k + 3(n^2 - n + 1) \qquad \text{(by the induction hypothesis)} \\ &= 3(k + n^2 - n + 1),\end{aligned}$$

showing that $3 \mid (n^3 + 2n)$, as required. ∎

3. (20 Points) Let x, y and n denote integers. Use induction to prove that

$$\forall n \geq 5: \ \exists x, y \geq 0 \ \text{ such that } \ n = 2x + 3y$$

(Hint: use strong induction form IId, with two base cases: $n = 5$ and $n = 6$.)

Proof:

I. We have $5 = 2 \cdot 1 + 3 \cdot 1$ and $6 = 2 \cdot 0 + 3 \cdot 2$, showing that the result is true when $n = 5$ and when $n = 6$.

II. Let $n > 6$ be chosen arbitrarily. Assume for all k in the range $5 \leq k < n$ that

$$\exists x_k, y_k \geq 0 \ \text{ such that } \ k = 2x_k + 3y_k.$$

We must show that

$$\exists x, y \geq 0 \ \text{ such that } \ n = 2x + 3y.$$

Observe $n > 6 \Rightarrow n \geq 7 \Rightarrow n - 2 \geq 5$. Therefore, by the induction hypothesis, there exist integers $x', y' \geq 0$ such that $n - 2 = 2x' + 3y'$. Let $x = x' + 1$ and $y = y'$. Then

$$\begin{aligned} n &= (n-2) + 2 \\ &= (2x' + 3y') + 2 \\ &= 2(x' + 1) + 3y' \\ &= 2x + 3y, \end{aligned}$$

as required. The result follows for all $n \geq 5$ by the 2nd PMI. ∎