CSE 16
Winter 2022
Quiz 3 **Solutions**

1. (15 Points) Determine the coefficient of x^{-2} in the full expansion of $\left(x+\frac{1}{x}\right)^{10}$.

   Solution:
   The Binomial Theorem gives

$$\left(x+\frac{1}{x}\right)^{10} = \sum_{k=0}^{10}\binom{10}{k}\cdot x^{10-k}\cdot\left(\frac{1}{x}\right)^{k} = \sum_{k=0}^{10}\binom{10}{k}\cdot x^{10-2k}$$

   Setting $10-2k=-2$, we obtain $2k=12 \Rightarrow k=6$. Therefore the coefficient of x^{-2} is

$$\binom{10}{6} = \frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2} = 10\cdot 3\cdot 7 = \mathbf{210}$$

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2. (15 Points) Determine the number of strings of length 10 from the alphabet $\{A, B, C\}$ that contain exactly 3 A's and 4 B's.

   Solution:
   Each such string must contain $10-3-4=3$ C's, and therefore is an anagram of $AAABBBBCCC$. Equivalently, they are permutations of the multiset $[A, A, A, B, B, B, B, C, C, C]$. By a theorem proved in class, we have

$$(\#\text{strings}) = \frac{10!}{3!\cdot 4!\cdot 3!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}{3\cdot 2\cdot 3\cdot 2} = 10\cdot 3\cdot 4\cdot 7\cdot 5 = \mathbf{4,200}$$

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3. (20 Points) Let $0 \leq k \leq n$. Show that $P(n,k) = P(n-1,k) + k \cdot P(n-1,k-1)$. You may use an algebraic or combinatorial proof.

Algebraic Proof:
RHS $= P(n-1,k) + k \cdot P(n-1,k-1)$

$$= \frac{(n-1)!}{\big((n-1)-k\big)!} + k \cdot \frac{(n-1)!}{\big((n-1)-(k-1)\big)!}$$

$$= \frac{(n-k)\cdot(n-1)!}{(n-k)\cdot(n-k-1)!} + \frac{k\cdot(n-1)!}{(n-k)!}$$

$$= \frac{(n-k)\cdot(n-1)!}{(n-k)!} + \frac{k\cdot(n-1)!}{(n-k)!}$$
$$= \frac{(n-k+k)\cdot(n-1)!}{(n-k)!}$$

$$= \frac{n\cdot(n-1)!}{(n-k)!}$$

$$= \frac{n!}{(n-k)!} = P(n,k) = \text{LHS}$$

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Combinatorial Proof:
The left hand side counts the number of k-permutations of an n-element set, or equivalently, the number of strings of length k from an alphabet of size n with no repeated symbols. We count the set another way by partitioning it into two disjoint classes of such strings.

Let the alphabet be $A = \{1,2,3,\dots,(n-1),n\}$, so that $A - \{n\} = \{1,2,3,\dots,(n-1)\}$ is an alphabet of size $(n-1)$. The two classes are:

I. Strings that **do not contain** n. These are all strings of length k from $A - \{n\}$, of which there are $P(n-1,k)$.

II. Strings that **do contain** n. We can form these strings by making the following choices in succession.

    (1) Choose a string of length $(k-1)$ from $A - \{n\}$: #ways $= P(n-1,k-1)$.

    (2) This string has k spaces between it's symbols. (e.g. "_a_b_c_d_" for $k = 5$). Choose one of those spaces in which to insert the symbol n. #ways $= k$.

    By the multiplication principle, the number of stings containing n is $k \cdot P(n-1,k-1)$.

By the addition principle, the number of strings in both classes is $P(n-1,k) + k \cdot P(n-1,k-1)$, which proves the above formula. ■