CSE 16
Homework Assignment 8
Solutions to Selected Problems

<u>Chapter 10 (page 195): 8</u>
Prove (by induction) that $\forall n \geq 1$:

$$\sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}.$$

Proof:
I. If $n = 1$, the above statement is $1/(1+1)! = 1 - 1/(1+1)!$, i.e. $1/2 = 1/2$, which is true.

II. Let $n \geq 1$ be chosen arbitrarily. Assume, for this n, that

$$\sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}.$$

We must show

$$\sum_{k=1}^{n+1} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+2)!}.$$

Observe that

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{k}{(k+1)!} &= \left(\sum_{k=1}^{n} \frac{k}{(k+1)!}\right) + \frac{n+1}{(n+2)!} \\
&= \left(1 - \frac{1}{(n+1)!}\right) + \frac{n+1}{(n+2)!} \qquad \text{(by the induction hypothesis)} \\
&= 1 - \frac{n+2}{(n+2)!} + \frac{n+1}{(n+2)!} \\
&= 1 - \left(\frac{(n+2)-(n+1)}{(n+2)!}\right) \\
&= 1 - \frac{1}{(n+2)!}.
\end{aligned}$$

The result follows for all $n \geq 1$ by the Principle of Mathematical Induction. ∎

Chapter 10 (page 195): 12

Prove (by induction) that $\forall n \geq 0$: $9|(4^{3n}+8)$.

Proof:

I. If $n = 0$, then $4^{3n} + 8 = 4^0 + 8 = 1 + 8 = 9$, hence $9|(4^{3\cdot 0}+8)$, establishing the base case.

II. Let $n \geq 0$ be chosen arbitrarily. Assume, for this n, that $9|(4^{3n}+8)$. Therefore $4^{3n} + 8 = 9k$ for some $k \in \mathbb{Z}$. We must show that $9|(4^{3(n+1)}+8)$. Observe

$$\begin{aligned} 4^{3(n+1)} + 8 &= 4^{3n+3} + 8 \\ &= 4^3 \cdot 4^{3n} + 8 \\ &= (63+1) \cdot 4^{3n} + 8 \\ &= 63 \cdot 4^{3n} + (4^{3n} + 8) \\ &= 9(7 \cdot 4^{3n}) + 9k && \text{By the induction hypothesis} \\ &= 9(7 \cdot 4^{3n} + k), \end{aligned}$$

showing that $9|(4^{3(n+1)}+8)$ as required.

The result follows for all $n \geq 0$ by the Principle of Mathematical Induction. ■

Chapter 10 (page 196): 26

Prove (by induction) that $\forall n \geq 1$: $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$, where F_n is the n^{th} Fibonacci number.

Proof:

Let $P(n)$ be the sentence $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$. We use weak induction form IIa.

I. $P(1)$ says $\sum_{k=1}^{1} F_k^2 = F_1 F_{1+1}$, i.e. $F_1^2 = F_1 F_2$, i.e. $1^2 = 1 \cdot 1$, or $1 = 1$, which is true.

II. Let $n \geq 1$. Assume $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$. We must show $\sum_{k=1}^{n+1} F_k^2 = F_{n+1} F_{n+2}$. We have

$$\begin{aligned} \sum_{k=1}^{n+1} F_k^2 &= \left(\sum_{k=1}^{n} F_k^2\right) + F_{n+1}^2 \\ &= (F_n F_{n+1}) + F_{n+1}^2 && \text{by the induction hypothesis} \\ &= F_{n+1}(F_n + F_{n+1}) \\ &= F_{n+1} F_{n+2} && \text{by the Fibonacci recurrence} \end{aligned}$$

It follows that $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$ for all $n \geq 1$ by the 1st PMI. ■