CSE 16
Homework Assignment 5
Solutions to Selected Problems

<u>(3.8) 8</u>
How many lists (x, y, z) of three integers are there with $0 \le x \le y \le z \le 100$?

Solution:
Each list corresponds to a 3-element multiset $[x, y, z]$ based on the 101-element set $\{0, 1, 2, 3, \ldots \ldots, 100\}$. By a formula derived in class the number of such lists is

$$\binom{3 + 101 - 1}{3} = \binom{103}{3} = \mathbf{176,851}. \qquad \blacksquare$$

Alternate Solution:
Given a list (x, y, z) satisfying $0 \le x \le y \le z \le 100$, define new variables $a = x, b = y - x, c = z - y$ and $d = 100 - z$. Then $a + b + c + d = 100$ where $a \ge 0, b \ge 0, c \ge 0, d \ge 0$. Thus, the required lists correspond exactly to the non-negative integer solutions (a, b, c, d) to $a + b + c + d = 100$. By an example covered in class, these solutions correspond to the 100-element multisets based on a 4-element set. The number of such lists is therefore

$$\binom{100 + 4 - 1}{100} = \binom{103}{100} = \binom{103}{3} = \mathbf{176,851} \qquad \blacksquare$$

<u>(3.9) 2</u>
You deal a pile of cards, face down, from a standard 52-card deck. What is the least number of cards the pile must have before you can be assured that it contains at least five cards of the same suit?

Solution:
Let n denote the number of cards in the pile. Since there are 4 suits in a standard deck, the pigeonhole principle (also called the *division principle* in the text), tells us to seek the least n for which

$$\left\lceil \frac{n}{4} \right\rceil = 5.$$

This is equivalent to the inequality $4 < n/4 \le 5$, which is the same as $16 < n \le 20$. The smallest such number is $\boldsymbol{n = 17}$. $\blacksquare$

<u>(3.10) 6</u>
Show that $\binom{3n}{3} = 3\binom{n}{3} + 6n\binom{n}{2} + n^3$.

Proof:
Let S be a set with $|S| = 3n$, and let A, B and C be subsets of S satisfying $A \cap B = A \cap C = B \cap C = \emptyset$, and $|A| = |B| = |C| = n$. The left hand side of the above formula, by its very definition, counts the number of 3-element subsets of S, i.e. the number of $\{x, y, z\} \subseteq S$, with x, y, z distinct. We count these same subsets in another way to obtain the right hand side.

Partition the 3-element subsets of S into 3 disjoint classes.

I. All three elements in the same set.

Choose $\{x, y, z\} \subseteq A$: #ways $= \binom{n}{3}$

Choose $\{x, y, z\} \subseteq B$: #ways $= \binom{n}{3}$

Choose $\{x, y, z\} \subseteq C$: #ways $= \binom{n}{3}$

Total number of sets of this type: $\binom{n}{3} + \binom{n}{3} + \binom{n}{3} = 3\binom{n}{3}$

II. Two elements in one set, one element in another set.

Choose $x \in A$ and $\{y, z\} \subseteq B$: #ways $= n\binom{n}{2}$

Choose $x \in A$ and $\{y, z\} \subseteq C$: #ways $= n\binom{n}{2}$

Choose $x \in B$ and $\{y, z\} \subseteq A$: #ways $= n\binom{n}{2}$

Choose $x \in B$ and $\{y, z\} \subseteq C$: #ways $= n\binom{n}{2}$

Choose $x \in C$ and $\{y, z\} \subseteq A$: #ways $= n\binom{n}{2}$

Choose $x \in C$ and $\{y, z\} \subseteq B$: #ways $= n\binom{n}{2}$

Total number of sets of this type: $6n\binom{n}{2}$

III. All three elements in different sets.

Choose $x \in A$, $y \in B$ and $z \in C$: #ways $= n \cdot n \cdot n = n^3$

Since every 3-element subset $\{x, y, z\} \subseteq S$ falls into exactly one of the above classes, the addition principle gives the number of such subsets as $3\binom{n}{3} + 6n\binom{n}{2} + n^3$, which proves the formula. ■