CSE 16
Homework Assignment 4
Solutions to Selected Problems

<u>(3.4) 16</u>

How many 4-permutations are there of the set $\{A, B, C, D, E, F\}$ if whenever A appears in the permutation, it is followed by E?

Solution:

This set of permutations partitions into two subsets: those containing A, and those not containing A. We deal with these two subsets separately, then use the addition principle.

(1) If the permutation does not contain A, then it is just a 4-permutation of the set $\{B, C, D, E, F\}$. The number of such permutations is $P(5,4) = 5 \cdot 4 \cdot 3 \cdot 2 = 120$.

(2) If the permutation does contain A, then since A must be followed by E, we can only place the A in positions $\{1, 2, 3\}$, as follows:

A E - -
- A E -
- - A E

After placing the E after the A, there are two positions left to be filled from the set $\{B, C, D, F\}$. This can be done in $P(4,2) = 4 \cdot 3 = 12$ ways. By the multiplication principle, the number of permutations containing A is $3 \cdot 12 = 36$.

By the addition principle, the total number of permutations is $120 + 36 = \mathbf{156}$. ■

(3.6) 10

Show that the formula $k\binom{n}{k} = n\binom{n-1}{k-1}$ is true for all integers n, k satisfying $0 \le k \le n$.

Solution:

The wording of the question does not specify if this is to be a combinatorial or an algebraic proof. We will accept either, and present both below.

Combinatorial Proof:

Suppose we have a group of n persons. In how many ways can we pick a committee consisting of k of these persons, with one member designated as the chair-person? We can solve this problem in two ways, as follows.

a. First pick a k-element subset of our group of n persons, then pick one of the k to be chair-person. By the multiplication principle these choices can be made in $k\binom{n}{k}$ ways, which is the left hand side of the equation.

But another procedure can also be used to count the number of committiees with chair-person.

b. First pick the chair-person from among the n people, then from the remaining $(n-1)$ persons, choose the other $(k-1)$ members of the committee. By the multiplication principle, this can be done in $n\binom{n-1}{k-1}$ ways, which is the right hand side of the equation.

Since both sides of the equation count the very same thing (the number of ways of picking a committee of size k with designated chair-person from a group of n persons), the two sides must be equal, i.e.

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

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Algebraic Proof:

$$\begin{aligned} k\binom{n}{k} &= k \cdot \frac{n!}{k!\,(n-k)!} \\ &= \frac{n!}{(k-1)!\,(n-k)!} \\ &= n \cdot \frac{(n-1)!}{(k-1)!\,(n-k)!} \\ &= n \cdot \frac{(n-1)!}{(k-1)!\,\big((n-1)-(k-1)\big)!} \\ &= n\binom{n-1}{k-1} \end{aligned}$$

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<u>(3.7) 4c</u>

How many 6-letter lists from the alphabet $\mathcal{S} = \{\mathrm{T, H, E, O, R, Y}\}$ (with repetition allowed) are there in which the sequence of letters (T, H, E) appears consecutively (i.e. in that order)?

Solution:

We define 4 sets of lists, based on the position of the sequence (T, H, E):

$$\begin{aligned} A &= \{\, (\mathbf{T}, \mathbf{H}, \mathbf{E}, X, Y, Z) \ : X, Y, Z \in \mathcal{S}\,\}, \\ B &= \{\, (X, \mathbf{T}, \mathbf{H}, \mathbf{E}, Y, Z) \ : X, Y, Z \in \mathcal{S}\,\}, \\ C &= \{\, (X, Y, \mathbf{T}, \mathbf{H}, \mathbf{E}, Z) \ : X, Y, Z \in \mathcal{S}\,\}, \\ D &= \{\, (X, Y, Z, \mathbf{T}, \mathbf{H}, \mathbf{E}) \ : X, Y, Z \in \mathcal{S}\,\}. \end{aligned}$$

The 4-set version of the Inclusion-Exclusion principle says

$$\begin{aligned} |A \cup B \cup C \cup D| = \ & |A| + |B| + |C| + |D| \\ & -|A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\ & +|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\ & -|A \cap B \cap C \cap D| \end{aligned}$$

Notice however that because of conflicts in the list positions 2, 3, 4 and 5, all but one of the 2-set intersections is empty. The only non-empty 2-set intersection is $A \cap D = \{\, (\mathrm{T, H, E, T, H, E}) \,\}$, containing exactly one list. As a consequence, all of the 3-set and 4-set intersections are empty as well. Observe also $|A| = |B| = |C| = |D| = 6^3 = 216$. It follows then that

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap D| = \mathbf{4 \cdot 216 - 1 = 863}.$$

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