cse 16 3-8-22

- SETs : evals
  deadline : Sunday 11:59 pm
  incentive : ?

- Quiz 5 : Thursday
- final : Wed. Mar. 16

Additional Problems in hw 9

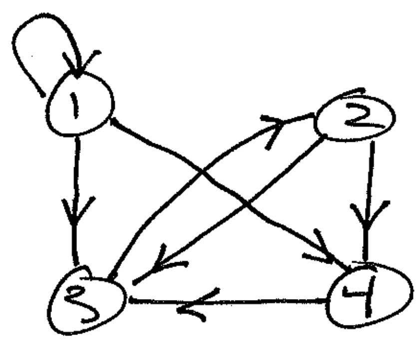
- how many relations on A if $|A| = n$.
  recall: $|A \times A| = n^2$, so
  $(\#\text{relations}) = 2^{n^2}$.

Matrix representation of a relation

Ex. $A = \{1, 2, 3, 4\}$ $n = 4$

$R = \{(1,1), (2,3), (3,2), (1,4), (4,3), (1,3), (2,4)\}$

digraph:



matrix:

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$$\boxed{M_{ij} = \begin{cases} 1 & iRj \\ 0 & i\not{R}j \end{cases}}$$

rule

(# $n \times n$ bit matrices) $= 2^{n^2}$

11.3 Equivalence Relations

Defn

A relation $R \subseteq A \times A$ on a set A is called an <u>equivalence relation</u> iff it is

(i) reflexive
(ii) symmetric
(iii) transitive

On $\mathbb{Z}$ we have 2 well known eq. relations

- $(=) = \{(x,x) \mid x \in \mathbb{Z}\}$
- $(\equiv_m) = \{(x,y) \mid x \equiv y \pmod{m}\}$

Proof that $\equiv_m$ is an eq. rel

- reflexive. let $x \in \mathbb{Z}$

$$m \mid 0 \Rightarrow m \mid (x-x) \Rightarrow x \equiv x \pmod{m}$$

- symmetric. let $x, y \in \mathbb{Z}$ also $a \in \mathbb{Z}$.

$m \mid a \Rightarrow m \mid (-a)$ since $a = km \Rightarrow -a = (-k)m$

therefore

$$\begin{aligned} x \equiv y \pmod{m} &\Rightarrow m \mid (x-y) \\ &\Rightarrow m \mid (y-x) \\ &\Rightarrow y \equiv x \pmod{m} \end{aligned}$$

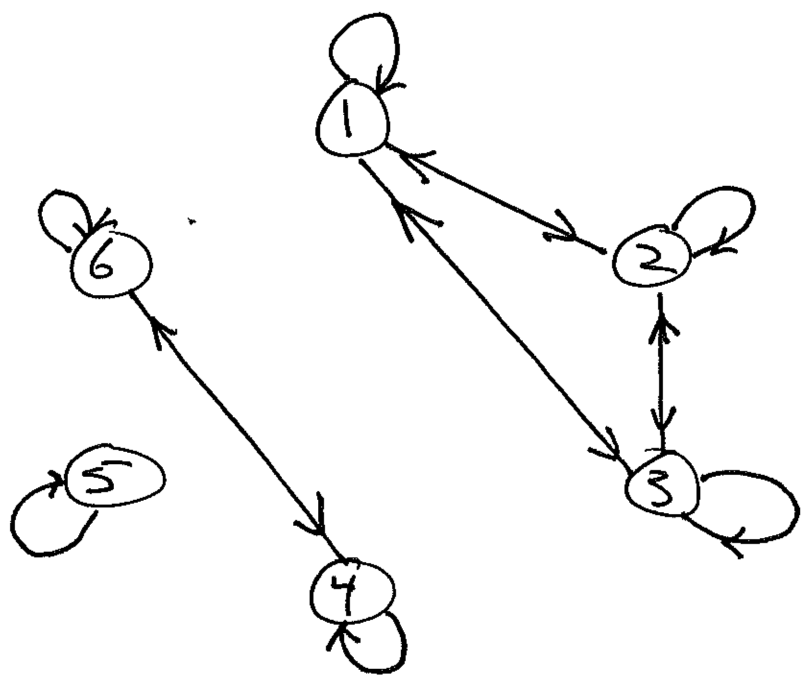
- transitive. last time.

$\blacksquare$

<u>Ex.</u> $A = \{1, 2, 3, 4, 5, 6\}$

$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3),$
$(3,1), (3,2), (3,3), (4,4), (4,6), (5,5),$
$(6,4), (6,6)\}$

digraph:



we see this <u>is</u> an eq. rel.

<u>Defn</u>

let $R \subseteq A \times A$ be an equivalence relation, and let $a \in A$. The <u>equivalence class containing a</u>, denoted $[a]$ is the set

$$[a] = \{x \in A \mid xRa\} \subseteq A$$

i.e. $[a]$ is the set of all $x \in A$ that are equivalent to a (under R).

<u>other notation</u>: $[a]_R$

Ex. (Previous)

$$[1] = [2] = [3] = \{1, 2, 3\}$$

$$[6] = [4] = \{4, 6\}$$

$$[5] = \{5\}$$

Lemma

Let R be an eq. rel. on A.

Then

$$a R b \quad \text{iff} \quad [a] = [b]$$

$\Rightarrow$ ✓

$\Leftarrow$ ✓

for all $a, b \in A$.

Proof.

($\Rightarrow$) suppose aRb. It's suff. to show $x \in [a] \Leftrightarrow x \in [b]$, for then $[a] = [b]$.

$$
\begin{aligned}
x \in [a] &\Rightarrow xRa && \text{(def.)}\\
&\Rightarrow xRa \wedge aRb && \text{(hyp.)}\\
&\Rightarrow xRb && \text{(trans.)}\\
&\Rightarrow x \in [b] && \text{(def.)}
\end{aligned}
$$

$$
\begin{aligned}
x \in [b] &\Rightarrow xRb && \text{(def.)}\\
&\Rightarrow xRb \wedge aRb && \text{(hyp.)}\\
&\Rightarrow xRb \wedge bRa && \text{(symm.)}\\
&\Rightarrow xRa && \text{(trans.)}\\
&\Rightarrow x \in [a] && \text{(def.)}
\end{aligned}
$$

$\therefore [a] = [b]$, as claimed.

($\Leftarrow$) suppose $[a]=[b]$. Since R is reflexive, we have aRa.

Thus

$$
\begin{aligned}
& a \in [a] && (\text{def.}) \\
\Rightarrow\ & a \in [b] && (\text{hyp.}) \\
\Rightarrow\ & aRb && (\text{def.})
\end{aligned}
$$

$\blacksquare$

<u>notice</u>: all 3 properties: reflexive, symmetric and transitive were used.

Given $m \in \mathbb{N}$, what are the eq. classes of $\equiv_m$?

By the lemma

$$x \in [a] \text{ iff } x \equiv_m a$$
$$\text{iff } x - a = km \quad (\text{some } k \in \mathbb{Z})$$
$$\text{iff } x = km + a$$

Thus

$$[a]_m = \{a + km \mid k \in \mathbb{Z}\}$$

Ex. $m = 2$

$$[0] = \{2k \mid k \in \mathbb{Z}\} = \{\text{even } \#\text{s}\}$$

$$[1] = \{2k+1 \mid k \in \mathbb{Z}\} = \{\text{odd } \#\text{s}\}.$$

$$[2] = \{2k+2 \mid k \in \mathbb{Z}\} = \{2(k+1) \mid k \in \mathbb{Z}\}$$
$$= \{2k \mid k \in \mathbb{Z}\} = [0]$$

so

$$[0] = [2] = [4] = \cdots = \{\text{even \#s}\}$$

$$[1] = [3] = [5] = \cdots = \{\text{odd \#s}\}$$

<u>Ex.</u> $m = 3$

$x \in [0]$ iff $x = 3k$

$x \in [1]$ " $x = 3k+1$

$x \in [2]$ " $x = 3k+2$

Thus

$$[0] = [3] = [6] = \cdots = \{3k \mid k \in \mathbb{Z}\}$$

$$[1] = [4] = [7] = \cdots = \{3k+1 \mid k \in \mathbb{Z}\}$$

$$[2] = [5] = [8] = \cdots = \{3k+2 \mid k \in \mathbb{Z}\}.$$

For general $m \in \mathbb{N}$, we can write any $x \in \mathbb{Z}$ as

$$x = km + r \qquad (0 \leq r < m).$$

This is the same as

$$x \in [r]$$

Thus

$$[r] = [m+r] = [2m+r] = \dots = \{km+r \mid k \in \mathbb{Z}\}.$$

The classes

$$[0], [1], [2], \dots, [m-1]$$

are <u>all</u> eq. classes of $\equiv_m$.

11.4 Eq. classes and Partitions

Recall: a Partition of a set A is a collection of subsets

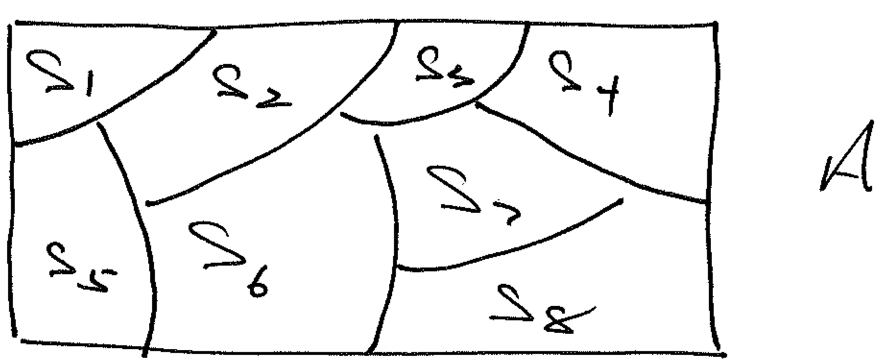
$$S_i \subseteq A \qquad (1 \le i \le k)$$

such that

(i) $S_i \neq \emptyset \qquad (1 \le i \le k)$

(ii) $S_i \cap S_j = \emptyset$ for $1 \le i < j \le k$

and (iii) $\bigcup_{i=1}^{k} S_i = A$



Theorem

Let R be an eq. rel. on a set A. Then the set of equivalence classes of R

$$\{[a] \mid a \in A\}$$

forms a partition of A.

Proof.

For any $a \in A$ we have $a \in [a]$ since aRa by reflexive prop. This shows $[a] \neq \emptyset$, which proves (i), similarly $\{a\} \subseteq [a]$, so

$$A = \bigcup_{a\in A} \{a\} \subseteq \bigcup_{a\in A} [a] \subseteq A$$

Therefore $A = \bigcup_{a\in A} [a]$, proving (iii).

Finally we show $[a] \neq [b] \Rightarrow [a] \cap [b] = \emptyset$ for any $a, b \in A$, establishing (ii)

we prove this by contraposition, i.e. we show

$$[a] \cap [b] \neq \emptyset \Rightarrow [a] = [b]$$

Assume $c \in [a] \cap [b]$. Then

$c \in [a] \Rightarrow cRa \Rightarrow aRc$ (symm.)

$c \in [b] \Rightarrow cRb$

$\therefore$ aRb (trans.)

$\therefore$ $[a]=[b]$ by prev. lemma.

we've shown

$$[a]\cap[b]\neq\phi \implies [a]=[b]$$

proving (ii). $\blacksquare$

In fact every partition of A arises in this way.

Theorem

Given a partition

$$P = \{ S_i \mid 1 \leq i \leq k \}$$

of the set A, there exists a relation R on A whose eq. classes are exactly the members S_i of P, i.e.

$$P = \{ [a]_R \mid a \in A \}.$$