CSE 16 3-3-22

Theorem

$\forall n \geq 1$: $\boxed{F_n = \frac{1}{\sqrt{5}}\left(\phi_1^n - \phi_2^n\right)}$

Proof.

I. $n=1$:

$$RHS = \frac{1}{\sqrt{5}}(\phi_1 - \phi_2) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)$$

$$= \frac{1}{\sqrt{5}}\left(\frac{\cancel{1}+\sqrt{5}-\cancel{1}+\sqrt{5}}{2}\right) = \frac{1}{\cancel{\sqrt{5}}}\left(\frac{\cancel{2}\cancel{\sqrt{5}}}{\cancel{2}}\right)$$

$$= 1 = F_1 = LHS \quad \checkmark$$

$n=2$:

$$RHS = \frac{1}{\sqrt{5}}\left(\phi_1^2 - \phi_2^2\right)$$

$$= \frac{1}{\sqrt{5}}\left((\phi_1 + \cancel{1}) - (\phi_2 + \cancel{1})\right) = \frac{1}{\sqrt{5}}(\phi_1 - \phi_2)$$

$$= 1 = F_2 = LHS \quad \checkmark$$

II. $\forall n > 2 : P(1) \wedge \ldots \wedge P(n-1) \Rightarrow P(n)$

Let $n > 2$. Assume for all k in the range $1 \le k < n$ that

$$F_k = \frac{1}{\sqrt{5}} \left(\phi_1^k - \phi_2^k \right).$$

we must show

$$F_n = \frac{1}{\sqrt{5}} \left(\phi_1^n - \phi_2^n \right).$$

so

$$\begin{aligned}
RHS &= \frac{1}{\sqrt{5}} \left(\phi_1^n - \phi_2^n \right) \\
&= \frac{1}{\sqrt{5}} \left(\phi_1^2 \cdot \phi_1^{n-2} - \phi_2^2 \cdot \phi_2^{n-2} \right) \\
&= \frac{1}{\sqrt{5}} \left((\phi_1 + 1) \phi_1^{n-2} - (\phi_2 + 1) \phi_2^{n-2} \right) \\
&= \frac{1}{\sqrt{5}} \left(\left(\phi_1^{n-1} + \phi_1^{n-2} \right) - \left(\phi_2^{n-1} + \phi_2^{n-2} \right) \right)
\end{aligned}$$

$$= \frac{1}{\sqrt{5}}\left(\left(\phi_1^{n-1} - \phi_2^{n-1}\right) + \left(\phi_1^{n-2} - \phi_2^{n-2}\right) \right)$$

$$= \frac{1}{\sqrt{5}}\left(\phi_1^{n-1} - \phi_2^{n-1}\right) + \frac{1}{\sqrt{5}}\left(\phi_1^{n-2} - \phi_2^{n-2}\right)$$

$$= F_{n-1} + F_{n-2}$$ { by the ind. hyp. with $k=n-1$, $k=n-2$.

$$= F_n$$ { by Fib. recurrence

$$= \text{LHS}$$ ∎

Exercise

- (ch. 10 #25)

$$\forall n \geq 1 : \sum_{k=1}^{n} F_k = F_{n+2} - 1$$

hint: weak ind.

• (Ch. 10 #29)

$$\forall n \geq 0: \sum_{k=0}^{n} \binom{n-k}{k} = F_{n+1}$$

hint: strong, 2 base cases

hint: use convention $\binom{a}{b} = 0$ when $b < 0$ or $a < b$.

Pascal's Δ:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

$\underline{n=5}$

$$\sum_{k=0}^{5} \binom{5-k}{k} = \binom{5}{0} + \binom{4}{1} + \binom{3}{2} + \cancel{\binom{2}{3}} + \cancel{\binom{1}{4}} + \cancel{\binom{0}{5}}$$

$$= 1 + 4 + 3 = 8 = F_6$$

Chapter 11 : Relations

11.1 General relations

Defn

A relation from a set A to a set B is a subset

$$R \subseteq \underbrace{A}_{\text{domain}} \times \underbrace{B}_{\text{codomain}}$$

notation given $x \in A$, $y \in B$ write

$$x R y \text{ to mean } (x,y) \in R$$

and

$$x \not R y \quad \text{"} \quad \text{"} \quad (x,y) \notin R$$

more generally we define a <u>relation</u> on $A_1, A_2, \ldots, A_n$ to be

$$R \subseteq A_1 \times A_2 \times \cdots \times A_n$$

used in theory of data bases.

<u>Defn</u>

A <u>relation on A</u> is a relation from A to A, i.e.

$$R \subseteq A \times A.$$

Relations on $\mathbb{Z}$:

- $<$ is the set $\{(x,y) \mid x<y\} \subseteq \mathbb{Z}\times\mathbb{Z}$
  so $(1,2) \in <$ and $(2,1) \notin <$.
- $\leq$ . . . . $x \leq y$ . . . . . .
- $>$ - - - - - - - - - - - -
- $\geq$ - - - - - - - - - - - -
- $=$ is set $\{(x,x) \mid x \in \mathbb{Z}\}$
- $\neq$ is the set $(\mathbb{Z}\times\mathbb{Z}) - \{(x,x) \mid x\in\mathbb{Z}\}$.
- $\mid$ (divisibility) is $\{(x,y) : x \mid y\}$
- for any $m \in \mathbb{N}$, $\equiv_m$ is the set
  $\{(x,y) : m \mid (x-y)\}$

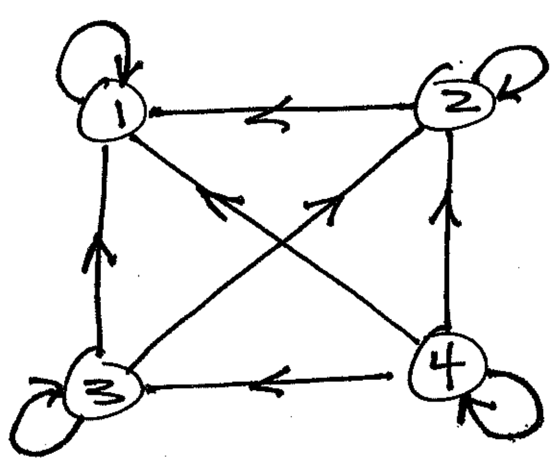
Relations on finite sets

<u>Ex.</u> $A = \{1, 2, 3, 4\}$ and

$$R = \{(1,1), (2,1), (2,2), (3,3), (3,2), (3,1), (4,4), (4,3), (4,2), (4,1)\}$$

Thus $4R1$, $3R2$ but $3 \not R 4$.

<u>directed graph</u> (<u>digraph</u>) representation.



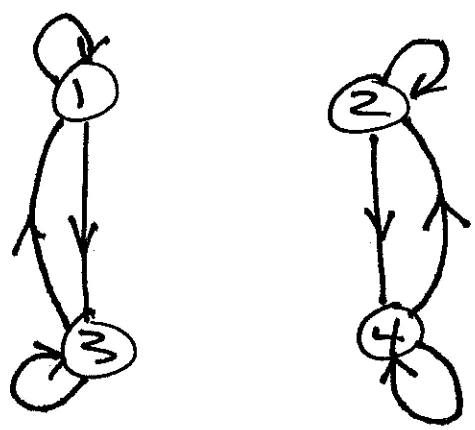
<u>note</u>: this is the $\geq$ relation on A.

$\boxed{\text{(x)} \longrightarrow \text{(y)} \text{ means } xRy}$

Ex. Again $A = \{1, 2, 3, 4\}$. let

$$S = \{(1,1), (1,3), (3,1), (3,3), (2,2), (2,4), (4,2), (4,4)\}$$

digraph:



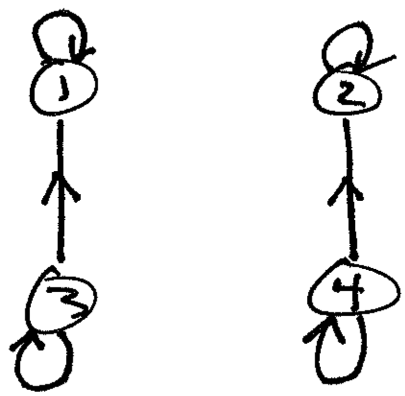
This is the 'same parity' relation on A. also

$$S = (\equiv_2)$$

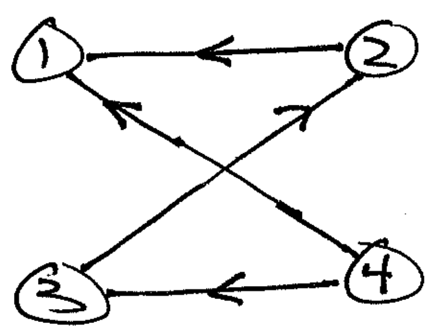
i.e. xSy iff $x \equiv y \pmod 2$.

Since Relations are sets, we can apply set operations to them.

$R \cap S$:



$R - S$:



<u>Exercise</u>: draw digraphs for

- $A \times A$
- ϕ
- $(A \times A) - S$
- $(A \times A) - R$
- $R \cup S$
- $R \oplus S$
- $S - R$

11.2 Properties of Relations

Defn

We call a relation $R \subseteq A \times A$

- **reflexive** iff $\forall x \in A$:

$$x R x$$

- **symmetric** iff $\forall x, y \in A$:

$$x R y \rightarrow y R x$$

- **transitive** iff $\forall x, y, z \in A$:

$$x R y \wedge y R z \rightarrow x R z$$

- **anti-symmetric** iff $\forall x, y \in A$

$$x R y \wedge y R x \rightarrow x = y.$$

check Relations on $\mathbb{Z}$.

Relation	Ref.	symm.	trans.	anti-symm
$<$	no	no	yes	yes
$\leq$	yes	no	yes	yes
$>$	no	no	yes	yes
$\geq$	yes	no	yes	yes
$=$	yes	yes	yes	yes
$\neq$	no	yes	no	no
$\mid$	yes	no	yes	yes
$\equiv_m$	yes	yes	yes	no

Exercise Prove all 24 entries in this table.

- $\equiv_m$ is transitive:

$$\left.\begin{array}{l} x \equiv_m y \rightarrow m \mid (x-y) \\ y \equiv_m z \rightarrow m \mid (y-z) \end{array}\right\} \rightarrow m \mid (x-y)+(y-z) \rightarrow m \mid (x-z) \rightarrow x \equiv_m z.$$

- $|$ is <u>not</u> symmetric.

we must show $\forall x,y : x|y \to y|x$

is false, i.e. that

$$\sim \forall x,y : x|y \to y|x$$

is true, i.e.

$$\exists x,y : x|y \wedge y \nmid x .$$

let $x=2$, $y=4$. Then $2|4$

but $4 \nmid 2$.

- $<$ is anti-symmetric.

we must show $\forall x,y$:

$$(x<y) \wedge (y<x) \to x=y$$

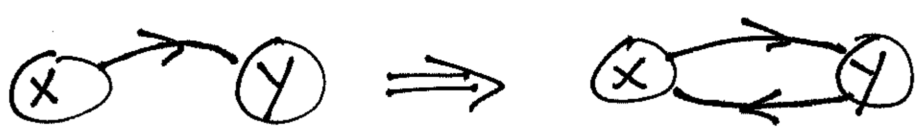
The hypothesis is a contradiction, so implication is vacuously true.

digraph interpretations

<u>reflexive</u>: xRx

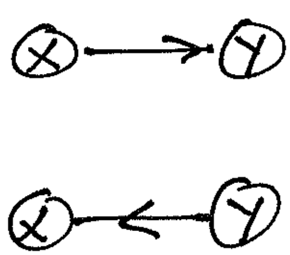
every vertex has a loop

<u>symmetric</u>: $xRy \rightarrow yRx$

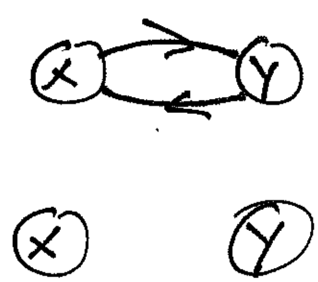


equivalently:

not possible



possible

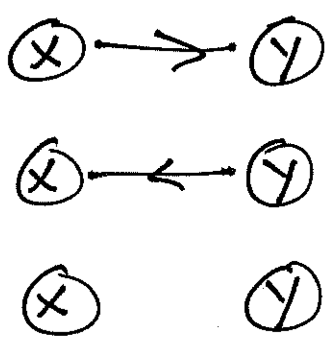


<u>transitive</u>: $xRy \wedge yRz \Rightarrow xRz$

<u>anti-symmetric</u>: $xRy \wedge yRx \Rightarrow x = y$

equivalently: given $x \neq y$

Possible



not Possible