cse 16 3-1-22

continue pf. of Bin. thm.

$$(x+y)^n = \vdots$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-k} y^k + \sum_{k=1}^{n} \binom{n-1}{k-1} x^{n-\cancel{1}-(k-\cancel{1})} \cdot y^{(k-\cancel{1})+\cancel{1}}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-k} y^k + \sum_{k=1}^{n} \binom{n-1}{k-1} x^{n-k} y^k$$

$$= \binom{n-1}{0} x^n y^0 + \sum_{k=1}^{n-1} \binom{n-1}{k} x^{n-k} y^k$$

$$+ \sum_{k=1}^{n-1} \binom{n-1}{k-1} x^{n-k} y^k + \binom{n-1}{n-1} x^0 y^n$$

$$= \binom{n-1}{0} x^n y^0 + \sum_{k=1}^{n-1} \left[\binom{n-1}{k} + \binom{n-1}{k-1} \right] x^{n-k} y^k + \binom{n-1}{n-1} x^0 y^n$$

$$= \binom{n}{0} x^n y^0 + \sum_{k=1}^{n-1} \binom{n}{k} x^{n-k} y^k + \binom{n}{n} x^0 y^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k,$$ as required.

result follows for all $n \geq 0$ by 1st PMI. ∎

Theorem

The prime factorization of every $n \geq 0$ is unique, i.e. if

$$p_1 \cdot p_2 \cdots p_k = n = q_1 \cdot q_2 \cdots q_\ell$$

where $p_1 \leq p_2 \leq \ldots \leq p_k$ and $q_1 \leq q_2 \leq \ldots \leq q_\ell$ are prime numbers, then $k = \ell$ and $p_i = q_i$ for $1 \leq i \leq k$.

Lemma (ch. 7 #30)

Let $a, b, p \in \mathbb{N}$ and suppose $a > 1$, $b > 1$ and p is prime. Then, if $p \mid a \cdot b$ then either $p \mid a$ or $p \mid b$.

Corollary

Let $a_1, a_2, \ldots, a_k \in \mathbb{Z}$ and suppose $p \mid (a_1 \cdot a_2 \cdots a_k)$, where p is prime.

Then $p \mid a_i$ for at least one i in range $1 \le i \le k$.

Proof of Theorem

Let $P(n) :=$ "the prime factorization of n is unique."

I. $P(1)$ is true since the empty product is unique.

II d. $\forall n > 1 : (P(1) \wedge \dots \wedge P(n-1)) \rightarrow P(n)$.

Let $n > 1$ be arbitrary. Assume the prime factorization of any integer less than n is unique. We must show the prime factorization of n is unique.

Suppose

$$(*) \qquad p_1 p_2 \cdots p_k = n = q_1 q_2 \cdots q_\ell$$

where $p_1 \le p_2 \le \dots \le p_k$, $q_1 \le q_2 \le \dots \le q_\ell$ are primes. Then

$$n = (p_1 \cdots p_{k-1}) \cdot p_k$$

so $p_k \mid n$, hence $p_k \mid q_1 \cdot q_2 \cdots q_\ell$.

By the corollary, $p_k \mid q_i$ for some $1 \leq i \leq l$. By re-indexing if necessary, we have $p_k \mid q_l$. But q_l has only $\{1, q_l\}$ as divisors. Since $p_k \neq 1$, we have $p_k = q_l$.

Cancel p_k & q_l on both sides of $*$, and define m by

$$p_1 p_2 \cdots p_{k-1} = m = q_1 q_2 \cdots q_{l-1}$$

$(p_1 \leq \cdots \leq p_{k-1},\ q_1 \leq \cdots \leq q_l)$. Since $p_k > 1$, we have $m < n$. By the ind. hyp., the prime factorization of m is unique.

Therefore $k-1 = \ell-1$ and

$$p_i = q_i \text{ for } 1 \le i \le k-1.$$

$\therefore$ $k = \ell$ and

$$p_i = q_i \text{ for } 1 \le i \le k.$$

Hence the prime factorization of n is unique. ∎

<u>strong induction with multiple base cases</u>:

To show $\forall n \ge n_0 : P(n)$, prove

I. $P(n_0), P(n_0+1), \ldots, P(n_1)$

IId. $\forall n > n_1 : (P(n_0) \wedge \ldots \wedge P(n-1)) \rightarrow P(n)$.

Ex

$$\forall n \geq 8 : \boxed{\underbrace{\exists x, y \geq 0 \text{ such that } n = 3x + 5y}_{P(n)}}$$

note: $P(1), \ldots, P(7)$ are not all true.

Proof.

I. $P(8) : 8 = 3 \cdot 1 + 5 \cdot 1$ ✓

$P(9) : 9 = 3 \cdot 3 + 5 \cdot 0$ ✓

$P(10) : 10 = 3 \cdot 0 + 5 \cdot 2$ ✓

II. $\forall n > 10 : (P(8) \wedge \ldots \wedge P(n-1)) \rightarrow P(n)$

let $n > 10$ be arbitrary. Assume for all k in range $8 \leq k < n$ that there exist $x_k, y_k \geq 0$ s.t.

$$k = 3x_k + 5 \cdot y_k$$

we must show that there exist $x, y \geq 0$ s.t.

$$n = 3x + 5y .$$

Since $n > 10 \Rightarrow n \geq 11 \Rightarrow n-3 \geq 8$, the ind. hyp. provides $x', y' \geq 0$ s.t.

$$n-3 = 3x' + 5y' .$$

Let $x = x'+1$ and $y = y'$. Then

$$\begin{aligned} n &= (n-3) + 3 \\ &= 3x' + 5y' + 3 \\ &= 3(x'+1) + 5y' \\ &= 3x + 5y . \end{aligned}$$

Result follows by 2^{nd} PMI.

Recursion

To define a sequence recursively:

- define a set of base object
- specify rule(s) for constructing new objects from existing ones.

Ex. Fibonacci Sequence.

$$\begin{cases} F_1 = 1 \\ F_2 = 1 \\ F_n = F_{n-1} + F_{n-2} \quad (\text{for } n \geq 3) \end{cases}$$

$$(F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8 \cdots)$$

$$= (1, 1, 2, 3, 5, 8, 13, 21, \ldots)$$

Ex. $\forall n \geq 1 : \boxed{\sum_{k=1}^{n} F_{2k-1} = F_{2n}} \leftarrow P(n)$

$n=1$: $F_1 = F_2$ ✓

$n=2$: $F_1 + F_3 = F_4$

$n=3$: $F_1 + F_3 + F_5 = F_6$

$\vdots$

Proof:

I. done ✓

IIa. $\forall n \geq 1 : P(n) \to P(n+1)$

let $n \geq 1$ be arbitrary. Assume

$$\sum_{k=1}^{n} F_{2k-1} = F_{2n} .$$

must show

$$\sum_{k=1}^{n+1} F_{2k-1} = F_{2(n+1)} .$$

so

$$\sum_{k=1}^{n+1} F_{2k-1} = \left(\sum_{k=1}^{n} F_{2k-1}\right) + F_{2(n+1)-1}$$

$$= F_{2n} + F_{2n+1} \quad \left\{ \begin{array}{l} \text{by the} \\ \text{ind hyp.} \end{array} \right.$$

$$= F_{2n+2} \quad \left\{ \begin{array}{l} \text{by the fib.} \\ \text{recurrence.} \end{array} \right.$$

$$= F_{2(n+1)} .$$

∎

<u>Exercise</u>

$$\forall n \geq 1: \sum_{k=1}^{n} F_k \cdot F_{k+1} = \begin{cases} F_{n+1}^2 & \text{if } n \text{ odd} \\ F_n F_{n+2} & \text{if } n \text{ even} \end{cases}$$

closed formula for F_n ?

i.e. not recursive ?

$$F_n = [\text{formula involving } n \text{ only}]$$

yes!!

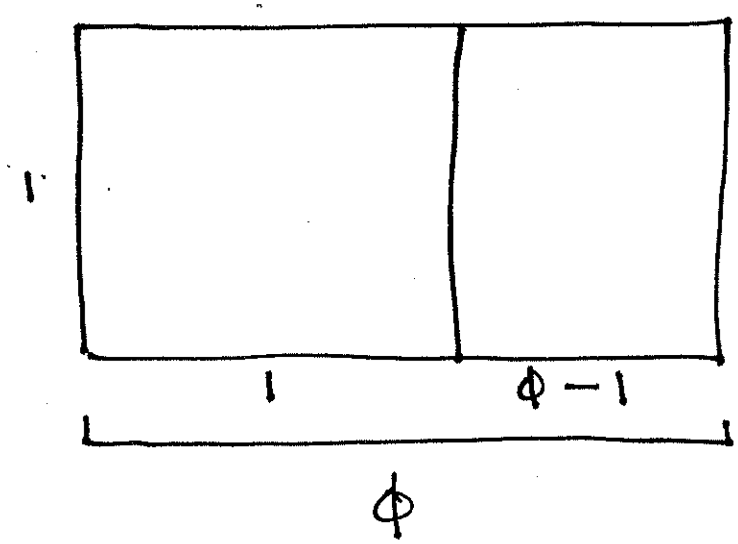
1st define

$$\begin{cases} \phi_1 = \dfrac{1+\sqrt{5}}{2} & \longleftarrow \text{Golden Ratio.} \\ \phi_2 = \dfrac{1-\sqrt{5}}{2} & \longleftarrow \text{Conjugate of G.R.} \end{cases}$$

note ϕ_1, ϕ_2 are roots of

$$(*) \qquad x^2 - x - 1 = 0$$

whats 'golden' about ϕ_1 ?



geometers thought the perfect rectangle would have aspect ratio ϕ satisfying

$$\frac{\phi}{1} = \frac{1}{\phi - 1}$$

$$\therefore \quad \phi(\phi - 1) = 1$$

$$\therefore \quad \phi^2 - \phi = 1$$

$$\therefore \quad \phi^2 - \phi - 1 = 0$$

i.e. $\boxed{\phi^2 = \phi + 1}$.

Both ϕ_1 & ϕ_2 satisfy this.

Thm

$P(n)$

$$\forall n \geq 1 : \boxed{F_n = \frac{1}{\sqrt{5}}\left(\phi_1^n - \phi_2^n\right)}$$

Proof. strong ind. 2 base cases

I

$$n=1: \text{RHS} = \frac{1}{\sqrt{5}}(\phi_1 - \phi_2)$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)$$

$$= \frac{1}{\sqrt{5}}\left(\frac{\cancel{1}+\sqrt{5}-\cancel{1}+\sqrt{5}}{2}\right)$$

$$= \frac{1}{\cancel{\sqrt{5}}} \cdot \frac{\cancel{2}\cancel{\sqrt{5}}}{\cancel{2}} = 1 = F_1 = \text{LHS} \checkmark$$

$n=2$ next time ....