An anagram of MISSISSIPPI can be thougt of as a permutation of 11-element multiset

$$[I, I, I, I, M, P, P, S, S, S, S]$$

based on the set

$$\{I, M, P, S\}$$

Theorem

The number of permutations of an n-element multiset based on a k-element set, with multiplicities $p_1, p_2, \ldots, p_k$ $(p_1 + p_2 + \cdots + p_k = n)$ is

$$\frac{n!}{p_1!\, p_2! \cdots p_k!}$$

Ex. MISSISSIPPI

$$\frac{11!}{4!\,1!\,2!\,4!} = \boxed{34{,}650}$$

Remark:

multinomial coefficient is given

$$\binom{n}{p_1, p_2, \ldots, p_k} = \frac{n!}{p_1!\,p_2!\cdots p_k!}$$

(where $p_1 + p_2 + \cdots + p_k = n$)

Theorem

$$\binom{n}{p_1, p_2, \ldots, p_k} = \binom{n}{p_1}\binom{n-p_1}{p_2}\binom{n-p_1-p_2}{p_3}\cdots\binom{n-p_1-p_2-\cdots-p_{k-1}}{p_k}$$

Exercise: Prove this
algebraic: easy
combinatorial: also easy !

Multinomial Theorem

Given $x_1, x_2, \ldots, x_k \in \mathbb{R}$, $n \in \mathbb{Z}$, $n \geq 0$:

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{p_1 + p_2 + \cdots + p_k = n} \binom{n}{p_1, p_2, \ldots, p_k} x_1^{p_1} x_2^{p_2} \cdots x_k^{p_k}$$

where the sum is over all k-tuples $(p_1, p_2, \ldots, p_k)$ of non-negative integers satisfying : $p_1 + p_2 + \cdots + p_k = n$

How many such terms are there ?

$$\text{ans} = \binom{n+k-1}{n} = \binom{n+k-1}{k-1}$$

One more multiset Problem:

Ex.

Find # of Pairs $(x,y) \in \mathbb{Z}\times\mathbb{Z}$ s.t.

$$0 \le x \le y \le 30.$$

This is same as # of non-neg. int. solns to

$$a+b+c=30$$

How? via the mapping

$$(x,y) \longleftrightarrow (a,b,c)$$

$$\begin{cases} a = x \\ b = y-x \\ c = 30-y \end{cases} \qquad \begin{cases} x = a \\ y = a+b \end{cases}$$

note
$0 \le x \to a \ge 0$
$x \le y \to b \ge 0$
$y \le 30 \to c \ge 0$

also:
$a \ge 0 \to 0 \le x$
$b \ge 0 \to x \le y$
$c \ge 0 \to y \le 30$

Thus:

$$\#(x,y) = \#(a,b,c)$$

$$= \binom{30+3-1}{30} = \binom{32}{30} = \boxed{496}$$

<u>Another soln</u>

By the sum rule

$$\#(0 \le x \le y \le 30) = \#(0 \le x < y \le 30) + \#(0 \le x \le 30)$$

$$= \binom{31}{2} + 31$$

$$= 465 + 31 = \boxed{496}$$

(3.9) Pigeonhole Principle

Defn

Let $x \in \mathbb{R}$.

- $\lfloor x \rfloor$ = greatest integer that is less than or eq. to x. called <u>floor</u> of x.

- $\lceil x \rceil$ = least int that is greater than or eq. to x. called ceiling of x

Alternately $\lfloor x \rfloor$ and $\lceil x \rceil$ are unique integers satisfying

- $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$

- $\lceil x \rceil - 1 < x \leq \lceil x \rceil$

equivalently: $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$

Pigeonhole Principle (P.P.)

Suppose n objects are placed in k boxes. Then

(a) at least one box contains $\lceil \frac{n}{k} \rceil$ (or more) objects.

(b) at least one box contains $\lfloor \frac{n}{k} \rfloor$ (or fewer) objects.

Proof:

The average number of objects per box is $\frac{n}{k}$.

We cannot have all boxes below this average. Thus some box must contain $\geq \frac{n}{k}$ objects.

Since the # of objects in a box must be an integer, we know some box contains $\geq \lceil \frac{n}{k} \rceil$ objects, proving (a)

Similarly, not all boxes contain above the average $\frac{n}{k}$ objects.

$\therefore$ some box contains $\leq \frac{n}{k}$ obj.

$\therefore$ some box contains $\leq \lfloor \frac{n}{k} \rfloor$, proving (b).

$\blacksquare$

Ex.

Find the smallest n such that in any group of n people, at least 10 were born in the same month.

By P.P. (a): if n objects (people) are placed in 12 boxes (months), then some box (month) contains $\lceil \frac{n}{12} \rceil$ (or more) objects (people).

We seek the smallest n s.t.

$$\left\lceil \frac{n}{12} \right\rceil = 10$$

$$\therefore \quad 9 < \frac{n}{12} \le 10$$

$$\therefore \quad 108 = 9 \cdot 12 < n \le 10 \cdot 12 = 120$$

$$\therefore \quad 108 < n \le 120$$

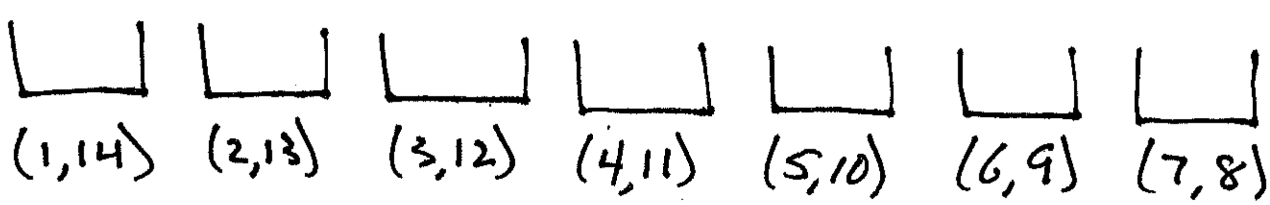
$$\therefore \quad n = 108 + 1 = \boxed{109}$$

Ex.

suppose 8 numbers are chosen from $\{1, 2, 3, \ldots, 14\}$. show that at least 2 of the 8 numbers must add up to 15.

we could check all $\binom{14}{8} = 3003$ 8-element ssets of $\{1, 2, \ldots, 14\}$ for the above property.

Instead form 7 labeled boxes



each of the 8 numbers are placed in the box with that number in the label.

By the P.P. (a), some box must contain 2 numbers, and those 2 numbers add up to 15.

Generalization

what is the smallest n s.t. any n-subset of

$$\{1, 2, 3, \ldots, 2k\}$$

contains 2 numbers that sum to $2k+1$.

answer: $n = k+1$

Exercise: explain this.

(3.10) Combinatorial Proofs

Ex. (3.6 #10)

show: $k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}$

Proof

Imagine we have .... see hw 4 solutions .....

Remark

more abstractly, both sides count the set

$$\{(x, A) \mid x \in A \subseteq S \text{ and } |A| = k\}$$

where S is a set satisfying $|S| = n$

Ex. (Generalization, also 3.10 #3)

Show

$$\binom{n}{k}\binom{k}{r}=\binom{n}{r}\binom{n-r}{k-r}$$

for $0 \le r \le k \le n$.

Proof.

Let $|S| = n$, we count the set

$$\{(A,B) \mid A \subseteq B \subseteq S,\ |A| = r,\ |B| = k\}$$

Two ways

I. (1) choose $B \subseteq S$ with $|B| = k$

ways $= \binom{n}{k}$

(2) choose $A \subseteq B$ with $|A| = r$

ways $= \binom{k}{r}$

By mult. princ. # pairs $= \binom{n}{k}\cdot\binom{k}{r} = LHS$.

II. (1) Choose $A \subseteq S$ with $|A| = r$.

$$\# \text{ ways} = \binom{n}{r}$$

(2) Choose $k-r$ elements from $S-A$ ($|S-A| = n-r$) and add them to A to form $B \subseteq S$ with $A \subseteq B$, and hence

$$|B| = r + (k-r) = k$$

$$\# \text{ ways} = \binom{n-r}{k-r}.$$

By the Product rule, the number of such pairs (A, B) is

$$\binom{n}{r} \cdot \binom{n-r}{k-r}.$$

Since I and II count the very same set, we have

$$\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}.$$