CSE 16 2-24-22

continue ...

observe

$$5(n+1)+8 = (5n+8)+5$$

$$< (n^2+4n+1)+5 \quad \left\{ \begin{array}{l} \text{by ind.} \\ \text{hyp.} \end{array} \right.$$

$$= n^2+4n+6$$

$$< n^2+4n+6+2n \quad \left\{ \begin{array}{l} \text{since } n \geq 4 \\ \Rightarrow 2n \geq 8 > 0 \end{array} \right.$$

$$= n^2+6n+6$$

$$\vdots$$

$$= (n+1)^2+4(n+1)+1 \quad \left\{ \begin{array}{l} \text{by some} \\ \text{algebra} \end{array} \right.$$

<u>Algebra</u>:

$$(n+1)^2+4(n+1)+1 = n^2+2n+1+4n+4+1$$

$$= n^2+6n+6$$

$$= (n^2+4n+6)+2n$$

∎

Ex. (ch. 10 #17)

Let $A_1, A_2, A_3, \ldots$ be an infinite seq. of subsets of some Universal set U.

$$\forall n \geq 1: \quad \boxed{\overline{\bigcap_{k=1}^{n} A_k} = \bigcup_{k=1}^{n} \overline{A_k}} \longleftrightarrow P(n)$$

(generalized de Morgan's law).

Proof.

I. $P(1)$ says: $\overline{\bigcap_{k=1}^{1} A_k} = \bigcup_{k=1}^{1} \overline{A_k}$,

i.e. $\overline{A_1} = \overline{A_1}$, which is true.

note: $P(2)$ says $\overline{\bigcap_{k=1}^{2} A_k} = \bigcup_{k=1}^{2} \overline{A_k}$, i.e.

$\overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2}$, which is deMorg. law.

II. $\forall n \geq 1 : P(n) \Rightarrow P(n+1)$.

let $n \geq 1$. Assume

$$\overline{\bigcap_{k=1}^{n} A_k} = \bigcup_{k=1}^{n} \overline{A_k}.$$

we must show

$$\overline{\bigcap_{k=1}^{n+1} A_k} = \bigcup_{k=1}^{n+1} \overline{A_k}.$$

so

$$\overline{\bigcap_{k=1}^{n+1} A_k} = \overline{\left(\bigcap_{k=1}^{n} A_k\right) \cap A_{n+1}}$$

$$= \left(\overline{\bigcap_{k=1}^{n} A_k}\right) \cup \overline{A_{n+1}} \quad \left\{ \text{by 2-set version of de Morg.} \right.$$

$$= \left(\bigcup_{k=1}^{n} \overline{A_k}\right) \cup \overline{A_{n+1}}$$

$$= \bigcup_{k=1}^{n+1} \overline{A_k} \qquad \blacksquare$$

<u>Exercise</u> (ch 10 #18)

$$\forall n \geq 1 : \quad \overline{\bigcup_{k=1}^{n} A_k} = \bigcap_{k=1}^{n} \overline{A_k}$$

<u>Exercise</u>

let $P_1, P_2, \ldots$ be an infinite seq. of state.

<u>notation</u>: $\displaystyle\bigvee_{k=1}^{n} P_k = P_1 \vee P_2 \vee \cdots \vee P_n$

$$\bigwedge_{k=1}^{n} P_k = P_1 \wedge P_2 \wedge \cdots \wedge P_n$$

Prove that

$$(1) \quad \sim\left(\bigwedge_{k=1}^{n} P_k\right) = \bigvee_{k=1}^{n} (\sim P_k)$$

and

$$(2) \quad \sim\left(\bigvee_{k=1}^{n} P_k\right) = \bigwedge_{k=1}^{n} (\sim P_k)$$

<u>Recall</u> PMI.

<u>Theorem</u>

Given any Prop. fcn. $P(n)$ on $\mathbb{N}$, the stmt.

$$\left[P(1) \wedge \forall n \left(P(n) \Rightarrow P(n+1)\right)\right] \Rightarrow \forall n\, P(n)$$

is true.

we use the W.O.P. of $\mathbb{N}$

<u>W.O.P.</u>

any non-empty subset of $\mathbb{N}$ contains a least element.

<u>Proof.</u>

Assume $P(1)$ and $\forall n\,(P(n) \to P(n+1))$ are both true. we must show

$$\forall n\, P(n)$$

is also true. let $S \subseteq \mathbb{N}$ be

$$S = \{n \in \mathbb{N} \mid \sim P(n)\}$$

i.e. S is the set of pos. ints for which $P(n)$ is false.

It is suff. to show $S = \phi$, for then $P(n)$ is true for <u>all</u> $n \in \mathbb{N}$.

Assume, to get a $\times$, that $S \neq \phi$.

Then, by the W.O.P., S contains a least element, call it m. Thus $m \in S$ but $m-1 \notin S$.

Since $P(1)$ is true, we have $1 \notin S$, so $m \neq 1$. Hence $m \geq 2$ and $m-1 \geq 1$. Thus $m-1 \in \mathbb{N}$.

Since $\forall n\,(P(n) \Rightarrow P(n+1))$ is true, we have, letting $n = m-1$, that

$$\boxed{P(m-1) \longrightarrow P(m)}$$

is true.

But recall $m-1 \notin S$, so

$$\boxed{P(m-1)}$$

is true. so by modus Ponens, we have that

$$P(m)$$

is true. But this implies $m \notin S$, which contradicts that $m \in S$.

This ✗ shows our assumption was false, hence $S = \phi$.

$\therefore \forall n\, P(n)$ is true. ∎

Variations on induction

I. $P(1)$

IIa. $\forall n \geq 1 : P(n) \to P(n+1)$.

IIb. $\forall n > 1 : P(n-1) \to P(n)$

IIb. is just a shift in the var. n.

Ex. $\forall n \geq 1 : \boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}$ ← $P(n)$ (again)

Proof:

I. same.

II b. $\forall n > 1 : P(n-1) \rightarrow P(n)$.

Let $n > 1$ be arbitrary. Assume

$$\sum_{k=1}^{n-1} k = \frac{n(n-1)}{2}.$$

we must show

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

so

$$\sum_{k=1}^{n} k = \left(\sum_{k=1}^{n-1} k \right) + n$$

$$= \left(\frac{n(n-1)}{2} \right) + n$$

$$= \frac{n(n-1) + 2n}{2}$$

$$= \frac{n(n-1+2)}{2}$$

$$= \frac{n(n+1)}{2}.$$

Result follows by PMI. ∎

<u>Exercise</u>

Do <u>all</u> prev. examples in form II b.

Both II a & II b are called <u>weak induction</u> or <u>1st PMI</u>.

<u>Another variation</u>

- <u>strong induction</u>
- <u>2nd PMI</u>

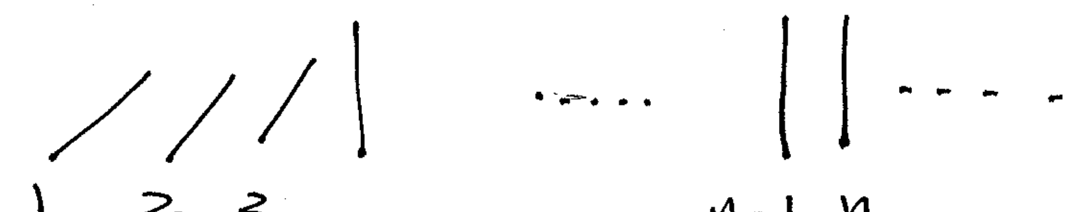
Goal: $\forall n\, P(n)$

I. Base: show $P(1)$

IIc. $\forall n \geq 1 : \underbrace{\left(P(1) \land P(2) \land \ldots \land P(n)\right)}_{\text{ind. hyp.}} \rightarrow \underbrace{P(n+1)}_{\text{ind. conc.}}$

IId. $\forall n > 1 : \underbrace{\left(P(1) \land P(n) \land \ldots \land P(n-1)\right)}_{\text{ind. hyp.}} \rightarrow \underbrace{P(n)}_{\text{ind. conc.}}$

IId.



assume dominos
1, 2, ..., n-1 fall

show n^{th}
domino falls

Theorem

Every positive integer can be expressed as the product of (zero or more) primes.

Rmk This is half of the F.T.A., the other half being uniqueness.

Proof:

we show

$$\forall n \geq 1 : \underbrace{\boxed{n \text{ is a product of primes}}}_{P(n)}$$

I. $P(1)$ is true since 1 is the empty product.

II d. $\forall n > 1: (P(1) \wedge P(2) \wedge \dots \wedge P(n-1)) \Rightarrow P(n)$

Let $n > 1$ be arbitrary. Assume for all k in the range $1 \le k < n$ that k is a product of primes.

We must show that n is a product of primes.

<u>case 1</u>: n is prime.

In this case, n is a product of one prime, itself.

<u>case 2</u>: n is composite

In this case $n = a \cdot b$ for some $a, b \in \mathbb{Z}$ with $1 < a < n$ and $1 < b < n$.

By the ind. hypothesis, both $P(a)$ and $P(b)$ are true, i.e a and b are products of primes.

$$a = p_1 p_2 \cdots p_\ell \qquad p_i \text{ prime } (1 \le i \le \ell)$$

$$b = q_1 q_2 \cdots q_m \qquad q_i \text{ prime } (1 \le i \le m)$$

Therefore

$$n = a \cdot b = (p_1 \cdots p_\ell) \cdot (q_1 \cdots q_m)$$

is also a prod. of primes.

Result follows for all $n \ge 1$ by 2^{nd} P.M.I.

<u>Theorem</u> (Binomial Thm)

let $x, y \in \mathbb{R}$.

$$\forall n \geq 0 : (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

<u>Algebraic Proof</u>

use Pascal's id: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ $(0<k<n)$.

I. $P(0)$ says

$$(x+y)^0 = \sum_{k=0}^{0} \binom{0}{k} x^{0-k} y^k$$

i.e. $1 = 1$.

II b. $\forall n>0: P(n-1) \Rightarrow P(n)$

Let $n>0$. Assume

$$(x+y)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-1-k} y^k .$$

must show

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Then

$$(x+y)^n = (x+y)(x+y)^{n-1}$$

$$= (x+y) \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-1-k} y^k$$

by the ind. hyp.

$$= x \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-1-k} y^{k} + y \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-1-k} y^{k}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-k} y^{k} + \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-1-k} y^{k+1}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-k} y^{k} + \sum_{k=1}^{n} \binom{n-1}{k-1} x^{n-k} y^{k}$$

$$\left\{ \begin{array}{l} \text{Replacing } k \text{ by } k-1 \\ 0 \le k \le n-1 \longrightarrow 0 \le k-1 \le n-1 \\ \qquad \longrightarrow 1 \le k \le n \end{array} \right\}$$

⋮

continue