

CSE 16 2-1-22

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PIE for 3 sets:

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad - |A \cap B| - |A \cap C| - |B \cap C| \\ &\quad + |A \cap B \cap C| \end{aligned}$$

PIE for 4 sets:

$$\begin{aligned} |A \cup B \cup C \cup D| &= |A| + |B| + |C| + |D| \\ &\quad - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| \\ &\quad \quad - |B \cap D| - |C \cap D| \\ &\quad + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| \\ &\quad \quad + |B \cap C \cap D| \\ &\quad - |A \cap B \cap C \cap D| \end{aligned}$$

Principle for n sets: $A_1, A_2, \dots, A_n$

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{i=1}^n |A_i|$$

$$- \sum_{1 \leq i < j \leq n} |A_i \cap A_j|$$

$$+ \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k|$$

$\vdots$

$$+ (-1)^{n+1} \left| \bigcap_{k=1}^n A_k \right|$$

Ex

Suppose $|A| = |B| = |C| = 20$,

$$|A \cup B \cup C| = 42, |A \cap B| = |A \cap C|,$$

and $B \cap C = \emptyset$. Determine $|A \cap B|$.

Let $x = |A \cap B|$. Then

$$42 = 3 \cdot 20 - x - x - 0 + 0$$

$$2x = 60 - 42 = 18$$

$$\boxed{x = 9}$$

(3.8) Multisets

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Defn

A multiset is an ordered collection of elements in which multiple memberships are allowed.

Defn

The cardinality of a multiset A , denoted $|A|$, is its # of members, including repetition

Ex. $A = [1, 1, 1, 2, 3, 3, 4, 4, 4]$

multiplicity of 1 : 3

" " 2 : 1

" " 3 : 2

" " 4 : 3

$$\therefore |A| = 3 + 1 + 2 + 3 = \boxed{9}$$

notation:

- sets $\{ \}$
- lists $()$
- multisets $[]$

Ex. let's produce multisets based on (i.e. using only elements from) $\{1, 2, 3\}$

	#
<u>Cardinality 0</u> : $[] = \emptyset$	$[]$
" " " " 1 : $[1], [2], [3]$	$[1], [2], [3]$
<u>Cardinality 2</u> :	
$[1, 1], [1, 2], [1, 3], [2, 2], [2, 3], [3, 3]$	$[6]$

cardinality 3!

#

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$[1, 1, 1], [1, 1, 2], [1, 1, 3], [1, 2, 2], [1, 2, 3]$

$[1, 3, 3], [2, 2, 2], [2, 2, 3], [2, 3, 3], [3, 3, 3]$

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cardinality 4!

exercise

$[1, 1, 1, 1], \dots, [3, 3, 3, 3]$

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Goal: develop a formula for the # of k -element multisets based on an n -element set.

we encode such a multiset as a string over alphabet

$\{*, 1\}$

look at case $n=3, k=4$.

multiset

string

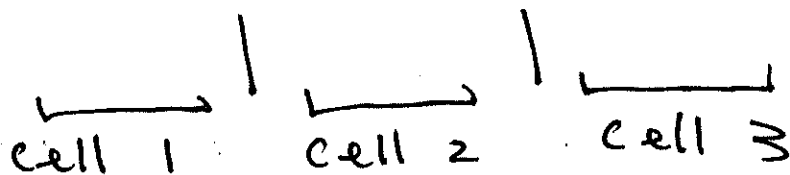
$[1, 2, 2, 3] \longleftrightarrow *|**|*$

$[1, 2, 3, 3] \longleftrightarrow *|*|**$

$[1, 3, 3, 3] \longleftrightarrow *||***$

$[3, 3, 3, 3] \longleftrightarrow ||****$

Each string is of length $6 = 2 + 4$,
and consists of 2 bars & 4 stars.
the 2 bars divide string into 3 cells:



cell 1 contains # stars = # 1's
 " 2 " " " = # 2's
 " 3 " " " = # 3's

note: encoding is invertible,
i.e. it's a one-to-one correspondence.

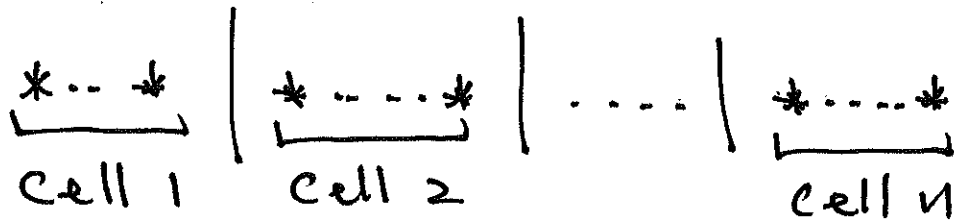
→ # of multisets of card. 4
based on a 3-set is same
as # of bit strings of len.
6 having exactly 2 1's

$$\binom{6}{2} = \frac{6 \cdot 5}{2} = \boxed{15}$$

In general, if

$$S = \{x_1, x_2, \dots, x_n\}$$

the k -element multisets based on S
are encoded as



Where # of *'s in cell i is the multiplicity of x_i in a multiset ($1 \leq i \leq n$)

observe:

$$(\# \text{ stars}) = \sum_{i=1}^n \text{mult}(x_i) = K$$

and

$$(\# \text{ bars}) = (\# \text{ cells}) - 1 = n - 1$$

Theorem

The # of k -element multisets based on an n -set is the # of strings on $\{*, | \}$ of length $(k+n-1)$ containing exactly k stars (or equivalently $(n-1)$ bars),

That is

$$\binom{k+n-1}{k} = \binom{k+n-1}{n-1}$$

Ex.

$$n=3, k=0 : \binom{0+3-1}{0} = \binom{2}{0} = 1$$

$$n=3, k=1 : \binom{1+3-1}{1} = \binom{3}{1} = 3$$

$$n=3, k=2 : \binom{2+3-1}{2} = \binom{4}{2} = 6$$

$$n=3, k=3 : \binom{3+3-1}{3} = \binom{5}{3} = 10$$

$$n=3, k=4 : \binom{4+3-1}{4} = \binom{6}{4} = 15$$

Ex.

$$n=4, k=6 : \binom{6+4-1}{6} = \binom{9}{6}$$

$$= \frac{\overset{3}{9} \cdot \overset{4}{8} \cdot 7}{\cancel{7} \cdot \cancel{8}}$$

$$= 4 \cdot 21 = \boxed{84}$$

Ex (P. 99, 3.8 #19)

Suppose you have an unlimited supply of Red, Blue & Green marbles. In how many ways can you select 20 marbles.

Each Collection of 20 marbles is a 20-element multiset based on the set $\{R, B, G\}$.

Thus $k=20$ and $n=3$, so there are

$$\binom{20+3-1}{20} = \binom{22}{20} = \boxed{231}$$

or

$$\binom{20+3-1}{3-1} = \binom{22}{2} = \boxed{231}$$

Ex. how many non-negative integer solutions to the equation

$$x + y + z = 20$$

Are there?

note: a solution is a triple (x, y, z) with $x, y, z \in \mathbb{Z}$, $x \geq 0$, $y \geq 0$, $z \geq 0$ satisfying above equation.

so, for example

$(3, 10, 7), (5, 0, 15)$ are solns

$(\frac{1}{2}, \frac{5}{2}, 17), (3, 10, 9)$ are not solns.

Observe: This is really the previous problem in disguise. How?

let $x = \# R's$

$y = \# G's$

$z = \# B's$

so has same answer: 231

for instance

solution multiset

$$(3, 10, 7) \leftrightarrow \left[\underbrace{RRR}_3 \underbrace{RR \dots R}_{10} \underbrace{GG \dots G}_7 \right]$$

$$\leftrightarrow \underbrace{***}_3 \mid \underbrace{** \dots *}_{10} \mid \underbrace{* \dots *}_7 \text{ (string)}$$

Ex

How many integer solns to

$$x + y + z = 20$$

satisfy : $x \geq 0, y \geq 0, z \geq 5$?

let $w = z - 5$. Then $w \geq 0$ and

$z = w + 5$. we have

$$x + y + z - 5 = 20 - 5 \quad (1)$$

$$x + y + w = 15 \quad (2)$$

count solutions (x, y, w) to (2)
satisfying $x \geq 0, y \geq 0, w \geq 0$. i.e.

$$K = 15, n = 3, \geq 0$$

$$(\# \text{solns}) = \binom{15+3-1}{15} = \binom{17}{15} = \boxed{136}$$

Permutations in which some
elements are indistinguishable

Defn An anagram of a word is
an arrangement of its letters,
where repeated letters are indistinguishable

Ex. anagrams of LOOK

KLOO	LOOK	OKLO	} 12
LKOO	KOOL	OLKO	
KOLO	OKOL	OOLK	
LOKO	OLOK	OOKL	

how to count without listing?

Temporarily make the 2 O's distinguishable.

$L O_1 O_2 K$

of Permutations of 4 symbols! $4!$

of ways to arrange O_1, O_2 ! $2!$

$L O_1 O_2 K \leftrightarrow L O_2 O_1 K$

$L O_1 K O_2 \leftrightarrow L O_2 K O_1$

Thus

$$\# \text{anag-ams} = \frac{4!}{2!} = \frac{4 \cdot 3 \cdot 2}{2} = \boxed{12}$$

Ex.

count anagrams of

MISSISSIPPI

(distinguishable) Permutations = 11!

Permutations of $I_1, I_2, I_3, I_4 = 4!$

" " " $S_1, S_2, S_3, S_4 = 4!$

" " " $P_1, P_2 = 2!$

so

$$\# \text{anag-ams} = \frac{11!}{4! \cdot 4! \cdot 2!} = \boxed{34,650}$$