cse 16 2-1-22

PIE for 3 sets:

$$|A \cup B \cup C| = |A| + |B| + |C|$$
$$- |A \cap B| - |A \cap C| - |B \cap C|$$
$$+ |A \cap B \cap C|$$

PIE for 4 sets:

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D|$$
$$- |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C|$$
$$- |B \cap D| - |C \cap D|$$
$$+ |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D|$$
$$+ |B \cap C \cap D|$$
$$- |A \cap B \cap C \cap D|$$

PIE for n sets: $A_1, A_2, \ldots, A_n$

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|$$

$$- \sum_{1 \le i < j \le n} |A_i \cap A_j|$$

$$+ \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k|$$

$$\vdots$$

$$+ (-1)^{n+1} \left|\bigcap_{k=1}^{n} A_k\right|$$

Ex

Suppose $|A| = |B| = |C| = 20$,

$|A \cup B \cup C| = 42$, $|A \cap B| = |A \cap C|$,

and $B \cap C = \phi$. Determine $|A \cap B|$.

Let $x = |A \cap B|$. Then

$$42 = 3 \cdot 20 - x - x - 0 + 0$$

$$2x = 60 - 42 = 18$$

$$\boxed{x = 9}$$

(3.8) Multisets

Defn

A multiset is an ordered collection of elements in which multiple memberships are allowed.

Defn

The cardinality of a multiset A, denoted $|A|$, is its # of members, including repetition

Ex. $A = [1, 1, 1, 2, 3, 3, 4, 4, 4]$

multiplicity of 1 : 3

" " 2 : 1

" " 3 : 2

" " 4 : 3

$\therefore |A| = 3 + 1 + 2 + 3 = \boxed{9}$

notation:

- sets $\{\ \}$
- lists $(\ )$
- multisets $[\ ]$

Ex. lets produce multisets based on (i.e. using only elements from)

$$\{1, 2, 3\}$$

		#
Cardinality 0 :	$[\,] = \emptyset$	$\boxed{1}$
" 1 :	$[1], [2], [3]$	$\boxed{3}$

Cardinality 2 :

$[1,1], [1,2], [1,3], [2,2], [2,3], [3,3]$ $\boxed{6}$

<u>cardinality 3:</u> #

[1,1,1], [1,1,2], [1,1,3], [1,2,2], [1,2,3]

[1,3,3], [2,2,2], [2,2,3], [2,3,3], [3,3,3] $\boxed{10}$

<u>cardinality 4:</u>

exercise....

[1,1,1,1], ..., [3,3,3,3] $\boxed{15}$

<u>Goal</u>: develop a formula for the # of k-element multisets based on an n-element set.

we encode such a multiset as a string over alphabet

$$\{ *, | \}$$

LT

look at case $n=3$, $k=4$.

<u>multiset</u>		<u>string</u>
[1, 2, 2, 3]	$\longleftrightarrow$	$* \mid * * \mid *$
[1, 2, 3, 3]	$\longleftrightarrow$	$* \mid * \mid * *$
[1, 3, 3, 3]	$\longleftrightarrow$	$* \mid \mid * * *$
[3, 3, 3, 3]	$\longleftrightarrow$	$\mid \mid * * * *$

Each string is of length $6 = 2+4$, and consists of 2 bars & 4 stars.
the 2 bars divide string into 3 cells:

$$\underbrace{\qquad\quad}_{\text{cell 1}} \mid \underbrace{\qquad\quad}_{\text{cell 2}} \mid \underbrace{\qquad\quad}_{\text{cell 3}}$$

cell 1 contains # stars = # 1's
" 2 " " " = # 2's
" 3 " " " = # 3's

<u>note</u>: encoding is invertible, i.e. its a one-to-one correspondence.

so # of multisets of card. 4 based on a 3-set is same as # of bit strings of len. 6 having exactly 2 1's

$$\binom{6}{2} = \frac{\overset{3}{\cancel{6}} \cdot 5}{\cancel{2}} = \boxed{15}$$

In general, if

$$S = \{x_1, x_2, \ldots, x_n\}$$

the k-element multisets based on S are encoded as

$$\underbrace{* \cdots *}_{\text{cell } 1} \;\Big|\; \underbrace{* \cdots *}_{\text{cell } 2} \;\Big|\; \cdots \;\Big|\; \underbrace{* \cdots *}_{\text{cell } n}$$

where # of $*$'s in cell i is the multiplicity of x_i in a multiset ($1 \le i \le n$)

observe:

$$(\#\text{stars}) = \sum_{i=1}^{n} \text{mult}(x_i) = k$$

and

$$(\#\text{ bars}) = (\#\text{cells}) - 1 = n-1$$

<u>Theorem</u>

The # of k-element multisets based on an n-set is the # of strings on $\{*, |\}$ of length $(k+n-1)$ containing exactly k stars (or equivalently $(n-1)$ bars).

That is

$$\binom{k+n-1}{k} = \binom{k+n-1}{n-1}$$

<u>Ex.</u>

$n=3,\ k=0$: $\binom{0+3-1}{0} = \binom{2}{0} = 1$

$n=3,\ k=1$: $\binom{1+3-1}{1} = \binom{3}{1} = 3$

$n=3,\ k=2$: $\binom{2+3-1}{2} = \binom{4}{2} = 6$

$n=3, k=3: \binom{3+3-1}{3} = \binom{5}{3} = 10$

$n=3, k=4: \binom{4+3-1}{4} = \binom{6}{4} = 15$

Ex.

$$n=4,\ k=6:\ \binom{6+4-1}{6} = \binom{9}{6}$$

$$= \frac{\cancel{9}^{3} \cdot \cancel{8}^{4} \cdot 7}{\cancel{3} \cdot \cancel{2}}$$

$$= 4 \cdot 21 = \boxed{84}$$

Ex (p. 99, 3.8 #19)

Suppose you have an unlimited supply of Red, Blue & Green marbles. In how many ways can you select 20 marbles.

Each Collection of 20 marbles is a 20-element multiset based on the set $\{R, B, G\}$.

Thus $k=20$ and $n=3$, so there are

$$\binom{20+3-1}{20} = \binom{22}{20} = \boxed{231}$$

or

$$\binom{20+3-1}{3-1} = \binom{22}{2} = \boxed{231}$$

__Ex.__ how many non-negative integer solutions to the equation

$$x+y+z=20$$

are there?

__note__: a solution is a triple (x, y, z) with $x, y, z \in \mathbb{Z}$, $x \geq 0$, $y \geq 0$, $z \geq 0$ satisfying above equation.

so, for example

$(3, 10, 7)$, $(5, 0, 15)$ are solns

$(\frac{1}{2}, \frac{5}{2}, 17)$, $(3, 10, 9)$ are not solns.

Observe: This is really the previous problem in disguise. How?

let $x = \#$ R's

$y = \#$ G's

$z = \#$ B's

so has same answer: $\boxed{231}$

for instance

<u>solution</u>	<u>multiset</u>
$(3,10,7)$	$\longleftrightarrow [\underbrace{RRR}_{3}\ \underbrace{BB\cdots B}_{10}\ \underbrace{GG\cdots G}_{7}]$

$$\longleftrightarrow \underbrace{***}_{3} \mid \underbrace{**\cdots *}_{10} \mid \underbrace{*\cdots *}_{7} \quad \text{(string)}$$

<u>Ex</u>

How many integer solns to

$$x + y + z = 20$$

satisfy: $x \geq 0$, $y \geq 0$, $z \geq 5$?

let $w = z - 5$. Then $w \geq 0$ and $z = w + 5$. We have

$$x + y + z - 5 = 20 - 5 \qquad (1)$$

$$x + y + w = 15 \qquad (2)$$

count solutions (x, y, w) to (2)

satisfying $x \geq 0,\ y \geq 0,\ w \geq 0$. i.e.

$k = 15,\ n = 3,$ so

$$(\#\text{solns}) = \binom{15+3-1}{15} = \binom{17}{15} = \boxed{136}$$

<u>Permutations in which some elements are indistinguishable</u>

<u>Defn</u> An <u>anagram</u> of a word is an arrangement of its letters, where repeated letters are indistingushible

Ex. anagrams of LOOK

KLOO	LOOK	OKLO
LKOO	KOOL	OLKO
KOLO	OKOL	OOLK
LOKO	OLOK	OOKL

$\boxed{12}$

how to count without listing?

Temporarily make the 2 O's distinguishable.

$$L\,O_1\,O_2\,K$$

# of permutations of 4 symbols: 4!

# of ways to arrange O_1, O_2: 2!

$$LO_1O_2K \longleftrightarrow LO_2O_1K$$

$$LO_1KO_2 \longleftrightarrow LO_2KO_1$$

Thus

$$\#\text{anagrams} = \frac{4!}{2!} = \frac{4\cdot 3\cdot \cancel{2}}{\cancel{2}} = \boxed{12}$$

Ex.

count anagrams of

MISSISSIPPI

(distinguishable) permutations = 11!

permutations of $I_1 I_2 I_3 I_4$ = 4!

" " " $S_1 S_2 S_3 S_4$ = 4!

" " " $P_1 P_2$ = 2!

so

$$\#\text{anagrams} = \frac{11!}{4!\,4!\,2!} = \boxed{34,650}$$