CSE 16 1-6-22

Note: as of today, we are meeting for lecture

$\boxed{11:30 \text{ am} \quad - \quad 1:05 \text{ pm}}$

(1.4) Power set of a set

Defn

The set of all subsets of a set S is called the power set of S

notation: $\mathbb{P}(S)$

Ex. $S = \{1, 2, 3\}$, $|S| = 3$

subsets of S: $|\mathbb{P}(S)| = 8 = 2^3$

$$\mathbb{P}(S) = \{ \phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}$$

Ex. $S = \{1, 2, 3, 4\}$, $|S| = 4$

$$\mathbb{P}(S) = \{ \phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\} \}$$

$|\mathbb{P}(S)| = 16 = 2^4$

<u>Ex.</u> $S = \{1, 2\}$, $|S| = 2$

$\mathbb{P}(S) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

$|\mathbb{P}(S)| = 4 = 2^2 = 2^{|S|}$

<u>Ex.</u> $T = \{\{1\}, \{1, 2\}\}$

$\mathbb{P}(T) = \{\emptyset, \{\{1\}\}, \{\{1,2\}\}, \{\{1\}, \{1,2\}\}\}$

<u>note</u>: $\{\{1\}\} \neq \{1\} \neq 1$

<u>note</u>: $\{1, 1, 1, 1, 2, 1, 1\} = \{1, 2\}$

Theorem

If S is a finite set, then so is $\mathbb{P}(S)$ and

$$|\mathbb{P}(S)| = 2^{|S|}$$

\# subsets

Ex. $\mathbb{P}(\emptyset) = \{\emptyset\}$ $\qquad 2^0 = 1$

$\mathbb{P}(\mathbb{P}(\emptyset)) = \mathbb{P}(\{\emptyset\})$

$= \{\emptyset, \{\emptyset\}\}$ $\qquad 2^1 = 2$

$\mathbb{P}(\mathbb{P}(\mathbb{P}(\emptyset))) = \mathbb{P}(\{\emptyset, \{\emptyset\}\})$

$= \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$ $\qquad 2^2 = 4$

what is :

$$|\mathbb{P}(\mathbb{P}(\mathbb{P}(\mathbb{P}(\emptyset))))| \overset{?}{=} 2^4 = 16$$

Exercise

find an expression for

$$\big|\underbrace{\mathbb{P}(\mathbb{P}(\cdots\cdots(\mathbb{P}}_{(\#\text{ of }\mathbb{P}\text{'s})=n}(\phi))\cdots))\big|$$

(1.5) set operations

Defn

let A, B be sets

- union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
- intersection: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- difference: $A - B = \{x \mid x \in A \text{ and } x \notin B\}$

$$= \{x \in A \mid x \notin B\}$$

Ex. $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$

- $A \cup B = \{1, 2, 3, 4, 5\}$
- $A \cap B = \{3, 4\}$
- $A - B = \{1, 2\}$
- $B - A = \{5\}$

note: • $A \cup B = B \cup A$
• $A \cap B = B \cap A$

• In general: $A - B \neq B - A$
unless $A = B$, in which case
$A - B = B - A = \emptyset$

Defn
we say A, B are disjoint iff

$$A \cap B = \emptyset$$

(1.6) Complements

Suppose all sets under discussion are subsets of some 'universal' set $\mathcal{U}$.

<u>Defn</u>

The <u>complement of A (relative to $\mathcal{U}$)</u> is

$$\overline{A} = \mathcal{U} - A$$
$$= \{x \in \mathcal{U} \mid x \notin A\}$$

<u>Ex</u> let $\mathcal{U} = \mathbb{N} = \{1, 2, 3, \dots\}$

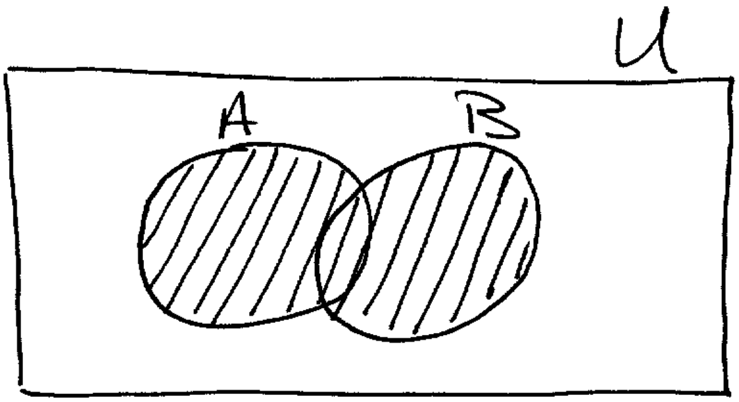
$$A = \{x \in \mathcal{U} \mid x \geq 8\} = \{8, 9, 10, \dots\}$$
$$\overline{A} = \{x \in \mathcal{U} \mid x < 8\} = \{1, 2, 3, 4, 5, 6, 7\}$$

Ex. let $U = \mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

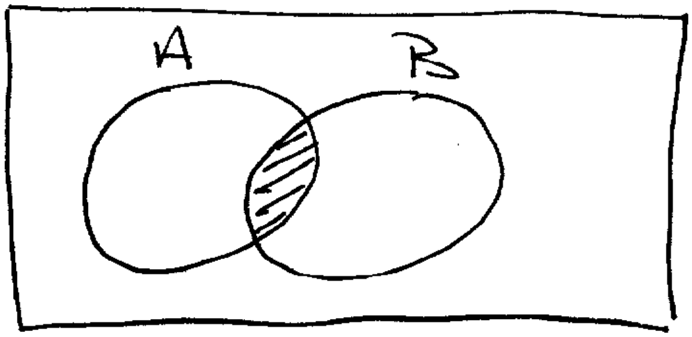
$$A = \{x \in U \mid x \geq 8\} = \{8, 9, 10, \dots\}$$

$$\bar{A} = \{x \in U \mid x < 8\} = \{\dots, -3, -2, -1, 0, 1, \dots, 7\}$$

(1.7) Venn Diagrams

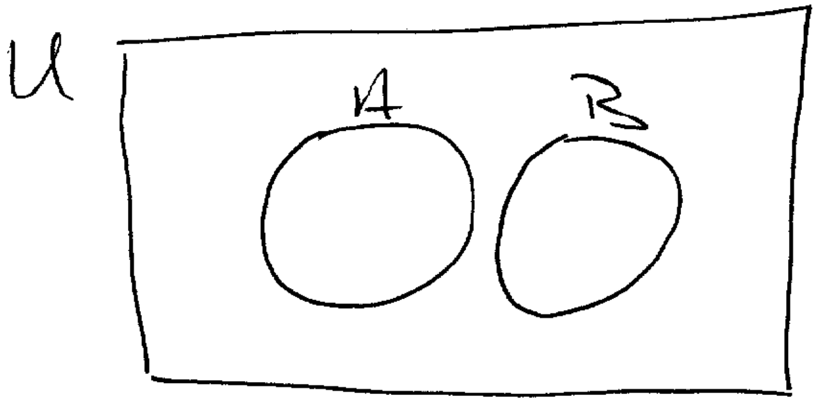


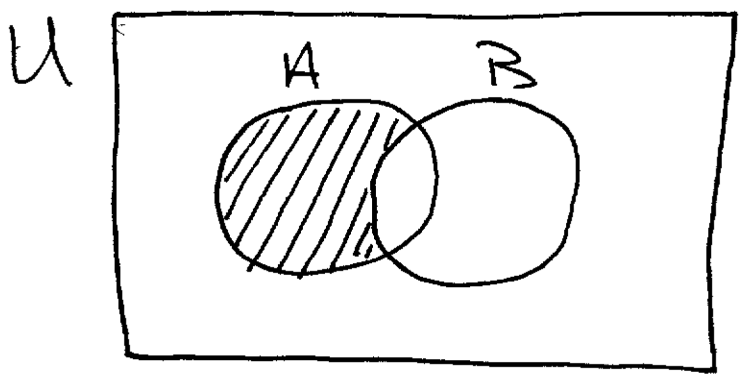
$A \cup B$



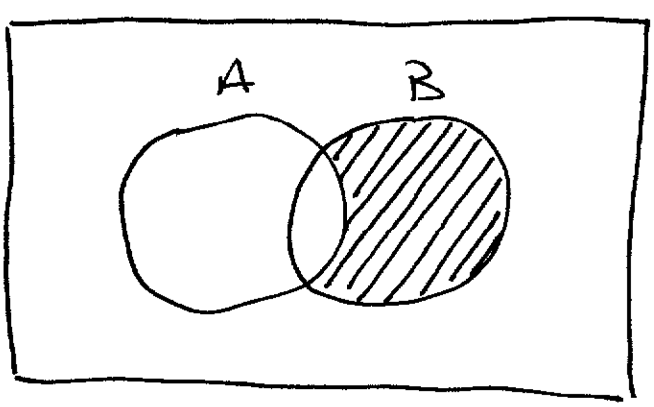
$A \cap B$

Disjoint sets: $A \cap B = \phi$

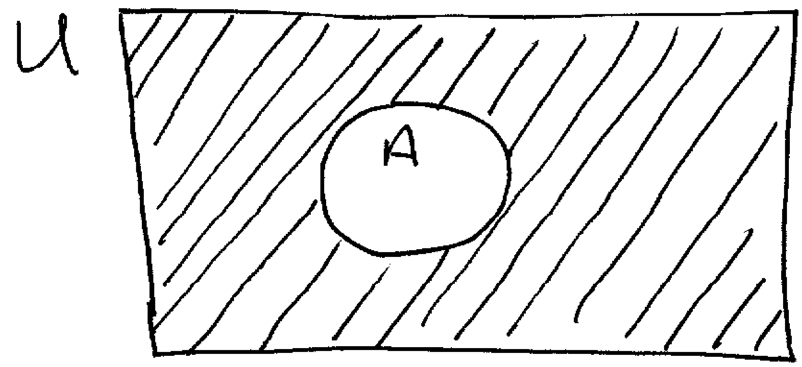




$A - B$



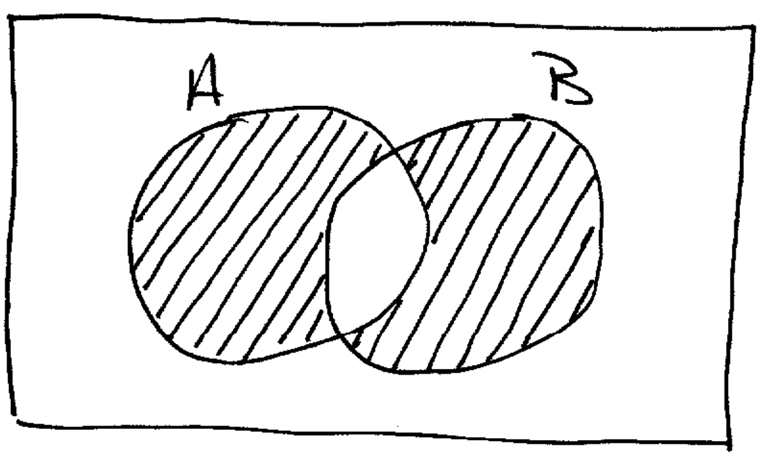
$B - A$



$\bar{A}$

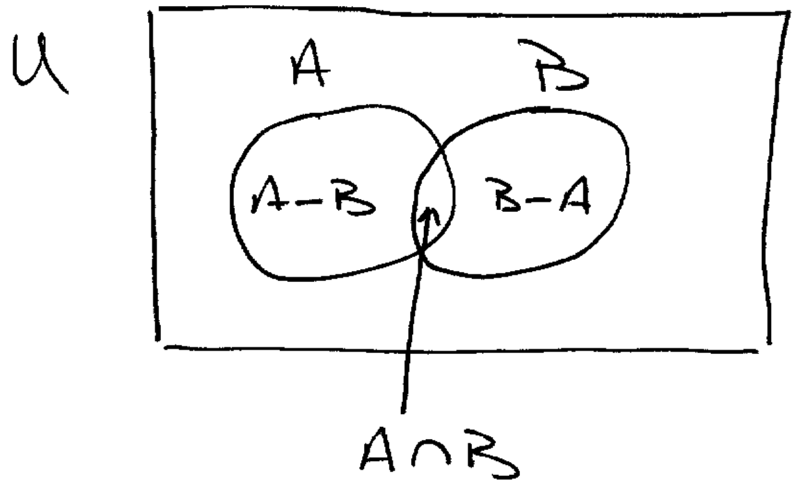
new set operation: symmetric difference

$$A \oplus B = (A-B) \cup (B-A)$$



$A \oplus B$

<u>note</u>: $A \cup B = (A-B) \cup (A \cap B) \cup (B-A)$



from this diagram we see that

$$A \oplus B = (A \cup B) - (A \cap B)$$

<u>note</u>: $A \oplus B = B \oplus A$

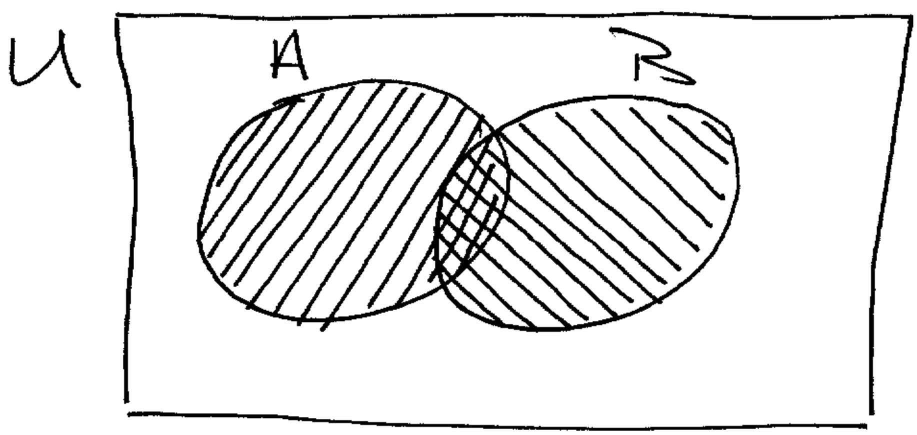
The same diagram shows:

Theorem

If A, B are finite sets, then so are $A \cup B$ and $A \cap B$, and

$$\boxed{|A \cup B| = |A| + |B| - |A \cap B|}$$

why?

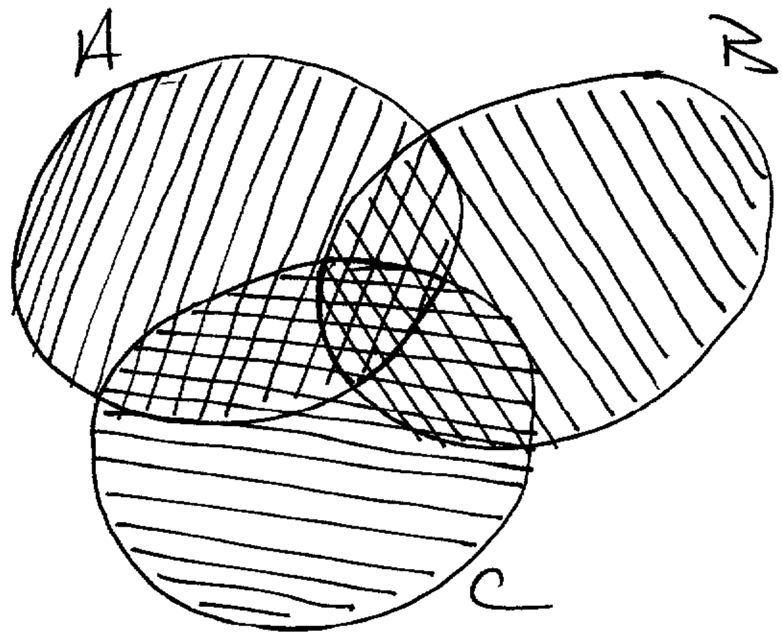


Also:

$$|A \cup B| = |A - B| + |B - A| + |A \cap B|$$

This is a special case of the principle of inclusion-exclusion (PIE)

3-set version:



$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Exercise: write the 4-set version, and draw a Venn Diagram.

(1.8) Indexed Sets

Let $A_1, A_2, A_3, \ldots\ldots\ldots$ be a (Possibly infinite) list of sets

notation

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n$$

$$= \{x \mid x \in A_i \text{ for } \underbrace{\text{at least one}}_{\text{some}} \ 1 \le i \le n\}$$

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots\cdots$$

$$= \{x \mid x \in A_i \text{ for } \underbrace{\text{at least one}}_{\text{some}} \ i \ge 1\}$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n$$
$$= \{x \mid x \in A_i \text{ for all } 1 \le i \le n\}$$

$$\bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap A_3 \cap \cdots\cdots$$
$$= \{x \mid x \in A_i \text{ for all } i \ge 1\}$$

we can pick out specific indices $i \in \mathbb{N}$ for both union & intersection

$$\bigcup_{i \in \{2,7,11\}} A_i = A_2 \cup A_7 \cup A_{11}$$

likewise for $\bigcap$

Suppose we have a set A_α for every $\alpha \in I$, where I is any (finite or infinite) set, called the <u>index set</u>.

$$\bigcup_{\alpha \in I} A_\alpha = \{x \mid x \in A_\alpha \text{ for } \underline{\text{some}}\ \alpha \in I\}$$

$$\bigcap_{\alpha \in I} A_\alpha = \{x \mid x \in A_\alpha \text{ for } \underline{\text{all}}\ \alpha \in I\}.$$