- quiz 2 today

<u>Recall</u>:

<u>Defn</u> $\binom{n}{k}$ = # k-subsets of an n-set

<u>Also</u> $\binom{n}{k}$ = # of len. n bit strings containing exactly k 1's.

$$\begin{array}{cccccccc} \underline{0} & \underline{1} & \underline{0} & \underline{0} & \cdots\cdots & \underline{1} & \underline{0} \\ 1 & 2 & 3 & 4 & & n-1 & n \end{array}$$

each bit string corresponds to a subset of $\{1, 2, 3, \ldots, n\}$

(3.6) Binomial Theorem

Recall:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad (0 \le k \le n)$$

we can comput:

n	
0	$\binom{0}{0} = 1$
1	$\binom{1}{0} = 1, \quad \binom{1}{1} = 1$
2	$\binom{2}{0} = 1, \binom{2}{1} = 2, \binom{2}{2} = 1$
3	$\binom{3}{0} = 1, \binom{3}{1} = 3, \binom{3}{2} = 3, \binom{3}{3}\ 1$
4	$\binom{4}{0} = 1, \binom{4}{1} = 4, \binom{4}{2} = 6, \binom{4}{3} = 4, \binom{4}{4} = 1$
$\vdots$	$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$

Pascal's Δ

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots
\end{array}
$$

Theorem (Pascal's identity)

$$(*) \qquad \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \qquad (1 \le k \le n)$$

$$(**) \qquad \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \qquad (0 < k < n)$$

Algebraic Proof of *

$$RHS = \binom{n}{k-1} + \binom{n}{k}$$

$$= \frac{n!}{(k-1)!\,(n-(k-1))!} + \frac{n!}{k!\,(n-k)!}$$

$$= \frac{n!}{(k-1)!\,(n-k+1)!} + \frac{n!}{k!\,(n-k)!}$$

$$= \frac{k \cdot n!}{k!\,(n-k+1)!} + \frac{(n-k+1)\cdot n!}{k!\,(n-k+1)!}$$

$$= \frac{k \cdot n! + (n-k+1)\,n!}{k!\,(n-k+1)!}$$

$$= \frac{(\cancel{k} + n - \cancel{k} + 1)\cdot n!}{k!\,((n+1)-k)!}$$

$$= \frac{(n+1)!}{k!\,((n+1)-k)!} = \binom{n+1}{k} = LHS.$$

Combinatorial Proof of **

Let $S = \{$ len. n bit strings containing exactly k 1's. $\}$

Then $|S| = \binom{n}{k}$. Also let

$$A = \{ x \in S \mid x \text{ ends in a } 1 \}$$

$$B = \{ x \in S \mid x \text{ ends in a } 0 \}$$

clearly $S = A \cup B$ and $A \cap B = \phi$.
By the addition principle

$$|S| = |A| + |B|$$

- every x in A can be obtained by taking a bit string of len. $(n-1)$ containing exactly $(k-1)$ 1's and appending 1 : $\underbrace{\_\ \_\ \_\ \_\ \cdots\ \_}_{n-1 \text{ with } (k-1) \text{ 1's}}\ \underline{1}$

this can be done in $\binom{n-1}{k-1}$ ways. $\therefore |A| = \binom{n-1}{k-1}$.

- every $x \in B$ can be obtained by taking a bit string of length n containing exactly k 1's, and appending 0.

$$\underbrace{\_\ \_\ \_\ \cdots\cdots\ \_}_{\substack{\text{len} = n-1 \\ \text{cont. } k \text{ 1's}}}\ \ 0$$

This can be done in $\binom{n-1}{k}$ ways.

$\therefore |B| = \binom{n-1}{k}$.

Hence

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Exercise

- write an alg. Proof of **
- " a comb. Proof of *

Another seq. of calculations that uses Pascal's Δ: let x, y be variables

n	
0	$(x+y)^0 = 1$
1	$(x+y)^1 = 1\cdot x + 1\cdot y$
2	$(x+y)^2 = 1\cdot x^2 + 2\cdot xy + 1\cdot y^2$
3	$(x+y)^3 = 1\cdot x^3 + 3\cdot x^2 y + 3\cdot xy^2 + 1\cdot y^3$
4	$(x+y)^4 = 1\cdot x^4 + 4\cdot x^3 y + 6\cdot x^2 y^2 + 4\cdot xy^3 + 1\cdot y^4$
	$\vdots$ $\qquad\qquad \vdots$

Evidently, when $(x+y)^n$ is fully expanded (and like terms combined) each term is of form

$$\binom{n}{k} x^{n-k} y^k \qquad (0 \le k \le n)$$

<u>Binomial Theorem</u>

for $n \geq 0$:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

wait for the <u>algebraic</u> proof until induction is covered.

Combinatorial Proof.

we number each factor in $(x+y)^n$

$$(x+y)^n = \underbrace{\overset{①}{(x+y)}\overset{②}{(x+y)}\overset{③}{(x+y)} \cdots\cdots \overset{ⓝ}{(x+y)}}_{n \text{ factors}}$$

when this product is fully expanded (before any like terms are combined) each term is a string of len. n over the alphabet $\{x, y\}$

Ex. $(x+y)^3 = \overset{①}{(x+y)}\overset{②}{(x+y)}\overset{③}{(x+y)}$

$$= xxx + xxy + xyx + xyy + yxx$$
$$+ yxy + yyx + yyy$$

i.e. we choose x or y from each of n factors, form product, then add terms any terms with the same # of x's (or same # of y's) can be combined.

So the coefficient of $x^{n-k} y^k$, after like terms are combined is the # of ways of choosing which k of n factors will contribute a y. Equivalently, this is the # of $\{x,y\}$-strings of len. n having exactly k y's. This # is $\binom{n}{k}$. So after like terms are combined, we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k .$$

$\blacksquare$

Ex. find the coeff. of $x^8 y^5$ in $(x+y)^{13}$

$$\text{ans} = \binom{13}{5} = \frac{13!}{5!\,8!}$$

$$= \frac{13 \cdot \cancel{12} \cdot 11 \cdot \cancel{10} \cdot 9}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}} = 13 \cdot 99 = \boxed{1,287}$$

Ex. find coeff. of x^5 in $(x-1)^{10}$

$$(x-1)^{10} = \sum_{k=0}^{10} \binom{10}{k} x^{10-k} (-1)^k$$

want $10 - k = 5 \therefore k = 10 - 5 = 5$

$$\text{ans} = \binom{10}{5} \cdot (-1)^5 = (-1) \cdot \frac{10!}{5!\,5!}$$

$$= -\frac{\cancel{10}^{\,} \cdot \cancel{9}^{3} \cdot \cancel{8}^{2} \cdot 7 \cdot 6}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}} = -3 \cdot 2 \cdot 7 \cdot 6$$

$$= \boxed{-252}$$

<u>Ex.</u> find coeff. of x^{-4} in $\left(x+\frac{1}{x}\right)^{20}$

$$\left(x+\frac{1}{x}\right)^{20} = \sum_{k=0}^{20} \binom{20}{k} x^{20-k} x^{-k}$$

$$= \sum_{k=0}^{20} \binom{20}{k} x^{20-2k}$$

$$20-2k = -4 \Rightarrow 24 = 2k \Rightarrow \boxed{k=12}$$

$$\text{ans} = \binom{20}{12} = \dots = \boxed{125{,}970}$$

<u>Ex.</u> let $x=1$, $y=-1$ in Bin. Thm.

$$0 = (1-1)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot 1^{n-k} \cdot (-1)^k$$

$$\therefore \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

we can obtain identities involving binomial coefficients by substituting values ~~for~~ x & y.

Ex. (3.6 #9)

Prove: $$9^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 10^{n-k}$$

Proof.

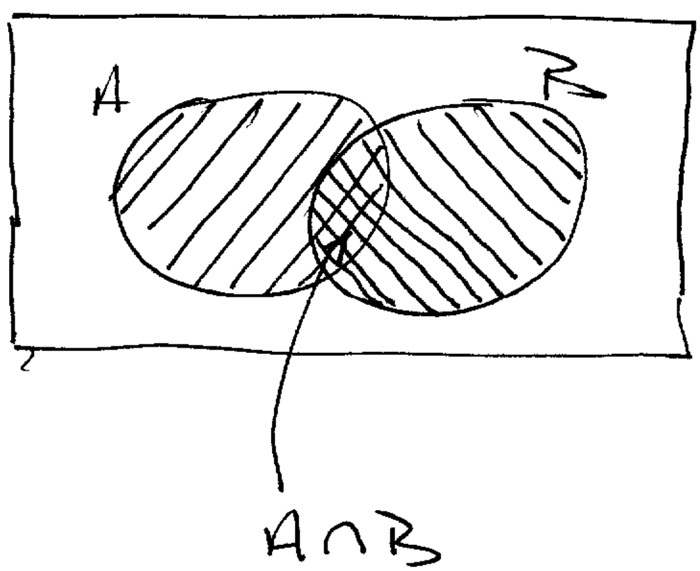
$$9^n = (10-1)^n = \sum_{k=0}^{n} \binom{n}{k} 10^{n-k} (-1)^k$$

∎

(3.7) Inclusion-Exclusion Principle

Recall: if A, B are finite sets, so are $A \cup B$ & $A \cap B$, and

$$|A \cup B| = |A| + |B| - |A \cap B|$$



Ex. how many 5-card Poker hands are there?

$$\binom{52}{5} = \frac{52!}{5!\,47!} = \boxed{2{,}598{,}960}$$

Ex. how many 5-card poker hands contain at least one red ace?

$A = \{\text{cont. } A\diamondsuit\}$

$B = \{\text{cont. } A\heartsuit\}$

$A \cap B = \{\text{cont. both } A\diamondsuit \text{ and } A\heartsuit\}$

$A \cup B = \{\text{cont. at least 1 red ace}\}$

$|A \cup B| = |A| + |B| - |A \cap B|$

$|A| = \binom{51}{4} = 249,900 = |B|$

$|A \cap B| = \binom{50}{3} = 19,600$

$\therefore |A \cup B| = |A| + |B| - |A \cap B|$

$= \boxed{480,200}$