CSE 16 1-25-22

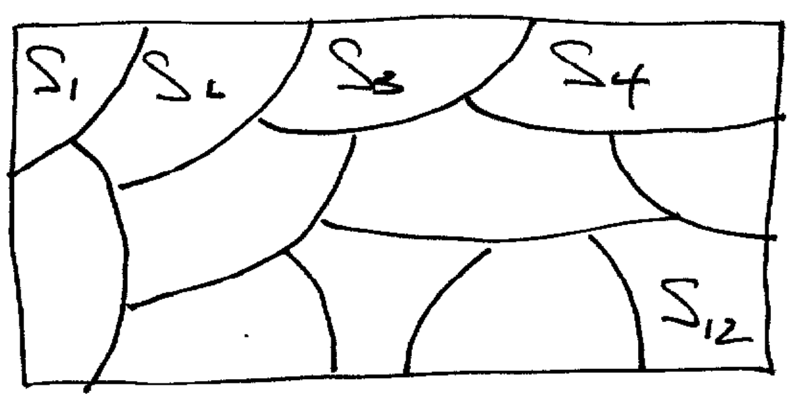
Defn

A partition of a set S is a collection of subsets $S_1, S_2, \ldots, S_n$ $(\neq \phi)$ of S such that

(1) $S_i \cap S_j = \phi$ for $1 \leq i < j \leq n$

(pairwise disjoint)

(2) $\bigcup_{i=1}^{n} S_i = S$

S 

Addition Principle

If $S_1, S_2, \ldots, S_n$ form a partition of of a finite set S, then

$$|S| = |S_1| + |S_2| + \cdots + |S_n|$$

(note: formula is correct even if some $S_i = \phi$.)

Ex. how many 4-digit numbers satisfy

- even (ends in even digit)
- no 0's, so only $\{1,2,3,4,5,6,7,8,9\}$
- exactly 1 5
- exactly 1 4, before the 5.

Partition the set of such strings into 3 subsets

		# strings
(1)	$\frac{4}{1}\ \frac{5}{1}\ \frac{\ }{7}\ \frac{\ }{3}$	21
		+
(2)	$\frac{4}{1}\ \frac{\ }{7}\ \frac{5}{1}\ \frac{\ }{3}$	21
		+
(3)	$\frac{\ }{7}\ \frac{4}{1}\ \frac{5}{1}\ \frac{\ }{3}$	21
		$\boxed{63}$

Observe: for any subset $A \subseteq S$ with $A \neq \phi$ and $A \subsetneq S$ we have a Partition into 2 subsets

$$A,\ S-A = \bar{A}$$

$$\therefore \quad |S| = |A| + |\bar{A}|$$

$$\therefore \quad |S-A| = |S| - |A|$$

Subtraction Principle

If S is finite and $A \subseteq S$, then

$$|S - A| = |S| - |A|$$

Ex. how many strings of length 6 from $\{a, b, c, d, e\}$ contain at least 1 d ?

let

$S = \{$strings of len 6 over $\{a, b, c, d, e\}\}$

$A = \{x \in S \mid x$ does not contain 'd' $\}$

note

$S - A = \{x \in S \mid x$ contains at least 1 d $\}$

Thus

$$|S-A| = |S| - |A|$$

$$= 5^6 - 4^6$$

$$= 15{,}624 - 4{,}096$$

$$= \boxed{11{,}529}$$

(3.4) Permutations

Defn

A *permutation* of a finite set S is an arrangement (list or string without repetition) of all of its elements

Ex. $S = \{1, 2, 3\}$

Permutations of S: #

123, 132, 213, 231, 312, 321 $\boxed{6}$

Ex $S = \{1, 2\}$

Permutations: 12, 21 $\boxed{2}$

Theorem

if $|S| = n$, then the # of Permutations of S is

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$$

Proof. follows from mult. Principle.

Defn

A <u>k-permutation</u> of a finite set S is an ordered arrangement of k of its elements.

<u>Ex.</u> $S = \{1, 2, 3\}$

		#
$k=3$:	123, 132, 213, 231, 312, 321	$\boxed{6}$
$k=2$:	12, 21, 13, 31, 23, 32	$\boxed{6}$
$k=1$:	1, 2, 3	$\boxed{3}$
$k=0$:	$\emptyset$	$\boxed{1}$

Notation

The # of k-permutations of an n-element set is denoted

$$P(n, k)$$

<u>observe</u>: $P(n,0) = 1$, $P(n,n) = n!$. Also if $k > n$, then $P(n,k) = 0$

By the mult. Principle

$$P(n,k) = \begin{cases} n(n-1)(n-2)\cdots(n-k+1) = \dfrac{n!}{(n-k)!} & 0 \le k \le n \\ 0 & k > n \end{cases}$$

<u>Ex.</u> list all $P(4,3) = 4.3.2 = \boxed{24}$

3-Permutations of $\{1, 2, 3, 4\}$.

1 2 3	2 1 3	3 1 2	4 1 2
1 3 2	2 3 1	3 2 1	4 2 1
1 2 4	2 1 4	3 1 4	4 1 3
1 4 2	2 4 1	3 4 1	4 3 1
1 3 4	2 3 4	3 2 4	4 2 3
1 4 3	2 4 3	3 4 2	4 3 2

Ex. (3.4 #7)

find # of 9-digit numbers satisfying

- no zeros : $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- no repetition
- all odd digits occur to left of all even digits

Solution:

odd digits $= \{1, 3, 5, 7, 9\}$
even digits $= \{2, 4, 6, 8\}$

(1) form a string of len. 5 from ODD dig.
(2) " " " " " 4 " even dig.
(3) concatenate 2 strings

(1) can be done in : $5!$ ways
(2) " " " " : $4!$ ways
(3) " " " " : 1 way

By mult. Principle

$$(\#\text{numbers}) = 5! \cdot 4! \cdot 1$$
$$= (120)(24)$$
$$= \boxed{2{,}880}$$

(3.5) Combinations

<u>Defn</u>

A <u>k-combination</u> of a finite set S is an unordered arrangement of k of its elements. Equivalently, a <u>k-combination</u> of S is just a k-element subset of S.

Notation

If $|S| = n$, the # of k-combinations of S is denoted

$$C(n, k) \quad \underline{\text{or}} \quad \binom{n}{k}$$

Read this: " n-choose-k "

Ex. $S = \{1, 2, 3\}$

		$C(3,k)$
$k=3$:	$\{1,2,3\}$	1
$k=2$:	$\{1,2\}, \{1,3\}, \{2,3\}$	3
$k=1$:	$\{1\}, \{2\}, \{3\}$	3
$k=0$:	$\emptyset$	1

Theorem

If $0 \leq k \leq n$ then

$$\binom{n}{k} = C(n,k) = \frac{n!}{k!\,(n-k)!}$$

Proof:
compute $P(n,k)$ again, in a different way. 2-step process to produce all k-permutations of an n-element set S.

(1) choose a k-element subset of S.
$\#\text{ways} = C(n,k)$

(2) choose a permutation of these k elements. $\#\text{ways} = k!$

By the mult. Principle

$$(\#\ k\text{-permutations of } S) = C(n,k)\cdot k!$$

i.e. $$P(n,k) = C(n,k)\cdot k!$$

By an earlier theorem

$$\frac{n!}{(n-k)!} = C(n,k)\cdot k!$$

Hence

$$C(n,k) = \frac{n!}{k!\,(n-k)!}.$$

$\blacksquare$

Ex.

how many bit strings of len. 10 contain exactly 6 ones?

Each bit string of len. 10 encodes a subset of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

1	0	1	0	1	1	0	1	1	0
1	$\boxed{2}$	3	$\boxed{4}$	5	6	$\boxed{7}$	8	9	$\boxed{10}$

Thus

$$1010110110 \longleftrightarrow \{1, 3, 5, 6, 8, 9\}$$

<u>notice</u>: $(\#\ 1\text{'s in bit string})$

$= (\text{size of subset}$

Thus

(# string of len. 10 containing exactly 6 1's)

$$= \binom{10}{6} = \frac{10!}{6!\,4!} = \frac{10 \cdot \cancel{9}^{3} \cdot \cancel{8} \cdot 7}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = \boxed{210}$$

Notice:

(# bit strings of len 10 with 6 1's)

= (# bit strings of len 10 with 4 0's)

Thus

$$\binom{10}{6} = \binom{10}{4}$$

choose which 6 positions are occupied by 1's

choose which 4 positions are occupied by 0's.

Indeed, both are equal to

$$\frac{10!}{6!\cdot 4!} = 210.$$

<u>Theorem</u>

If $0 \le k \le n$, then

$$\binom{n}{k} = \binom{n}{n-k}$$

<u>Algebraic Proof</u>:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$= \frac{n!}{(n-k)!\,(n-(n-k))!}$$

$$= \binom{n}{n-k} \quad ■$$

Combinatorial Proof

The LHS counts # of bit strings of len. n with exactly k 1's.

The RHS counts # of bit strings of len n with exactly $n-k$ 0's.

Thus LHS & RHS count the very same set of strings.

$\therefore$ LHS = RHS

$$\therefore \quad \binom{n}{k} = \binom{n}{n-k}$$

∎