cse 16 1-18-22

Ex. (logical equiv.)

$$p \to q \equiv \sim q \to \sim p \quad \checkmark$$

p	q	$p \to q$	$\sim q$	$\sim p$	$\sim q \to \sim p$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	1	0	0	1

✓

contrapositive law

$$P \Rightarrow Q \;=\; \sim Q \Rightarrow \sim P$$

DeMorgan's laws

$$\sim(P \wedge Q) \;=\; \sim P \vee \sim Q$$
$$\sim(P \vee Q) \;=\; \sim P \wedge \sim Q$$

Commutative laws

$$P \wedge Q \;=\; Q \wedge P$$
$$P \vee Q \;=\; Q \vee P$$

Distributive laws

$$P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$$
$$P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$$

Associative laws

$$P \wedge (Q \wedge R) = (P \wedge Q) \wedge R$$
$$P \vee (Q \vee R) = (P \vee Q) \vee R$$

Implication Law

$$P \Rightarrow Q \;=\; \sim P \vee Q$$

we can use these identities to prove other identities algebraically.

Ex. $(P \vee Q) \Rightarrow R = (P \Rightarrow R) \wedge (Q \Rightarrow R)$

$$
\begin{aligned}
(P \vee Q) \Rightarrow R &= \sim(P \vee Q) \vee R && (\text{imp.}) \\
&= (\sim P \wedge \sim Q) \vee R && (\text{DeMorg.}) \\
&= (\sim P \vee R) \wedge (\sim Q \vee R) && (\text{dist. \& com.}) \\
&= (P \Rightarrow R) \wedge (Q \Rightarrow R) && (\text{imp.})
\end{aligned}
$$

other identities

__Domination__

$$P \vee T = T$$
$$P \wedge F = F$$

__Identity__

$$P \wedge T = P$$
$$P \vee F = P$$

Idempotent

$$P \vee P = P$$
$$P \wedge P = P$$

Double Negation

$$\sim(\sim P) = P$$

Absorption

$$P \vee (P \wedge Q) = P$$
$$P \wedge (P \vee Q) = P$$

Negation

$$P \vee \sim P = T$$
$$P \wedge \sim P = F$$

Exercise

Prove all of these logical identities using truth tables.

Ex show $(p \wedge q) \to (p \to q)$ is a tautology. ✓

$$(p \wedge q) \to (p \to q)$$

$= \sim(p \wedge q) \vee (\sim p \vee q)$ (imp.)

$= (\sim p \vee \sim q) \vee (\sim p \vee q)$ (De Morg)

$= (\sim p \vee \sim p) \vee (\sim q \vee q)$ (com., assoc.)

$= \sim p \vee T$ (idem., negation)

$= T$ (domination)

Exercise

show $\sim(p \to q) \to p$ is a tautology both algebraically and with truth tables.

Exercise:

Determine whether

$$(p \vee \sim q) \wedge (q \vee \sim r) \wedge (r \vee \sim p)$$

is satisfiable. (ans: yes).
find assignments that make it true.

Set Theory and logic

Let $A, B, C \subseteq U$. Then

- $A \cap B = \{x \in U \mid (x \in A) \wedge (x \in B)\}$
- $A \cup B = \{x \in U \mid (x \in A) \vee (x \in B)\}$
- $A \oplus B = \{x \in U \mid (x \in A) \oplus (x \in B)\}$

- $\overline{A} = \{x \in U \mid \sim(x \in A)\}$
  $= \{x \in U \mid x \notin A\}$

- $\phi = \{x \in U \mid F\}$

- $U = \{x \in U \mid T\}$

<u>analogy</u>

sets	$\longleftrightarrow$	statements
$\cap$	$\longleftrightarrow$	$\wedge$
$\cup$	$\longleftrightarrow$	$\vee$
$\oplus$	$\longleftrightarrow$	$\oplus$
$-$ (complement)	$\longleftrightarrow$	$\sim$ (negation)
ϕ	$\longleftrightarrow$	F
U	$\longleftrightarrow$	T

Ex DeMorgans laws for sets

(1) $\overline{A \cup B} = \overline{A} \cap \overline{B}$ ✓

(2) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Proof of (1)

$$\overline{A \cup B} = \{x \in U \mid \sim (x \in A \cup B)\}$$
$$= \{x \in U \mid \sim ((x \in A) \vee (x \in B))\}$$
$$= \{x \in U \mid \sim (x \in A) \wedge \sim (x \in B)\}$$
$$= \{x \in U \mid (x \in \overline{A}) \wedge (x \in \overline{B})\}$$
$$= \overline{A} \cap \overline{B}$$

Exercise: Prove (2) same way.

Every logical identity has an analog in set theory.

DeMorgan's laws

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$
$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

commutative laws

$$A \cap B = B \cap A$$
$$A \cup B = B \cup A$$

Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Associative Laws

$$A \cap (B \cap C) = (A \cap B) \cap C$$
$$A \cup (B \cup C) = (A \cup B) \cup C$$

Domination laws

$$A \cup \mathcal{U} = \mathcal{U}$$
$$A \cap \emptyset = \emptyset$$

Identity laws

$$A \cap U = A$$
$$A \cup \emptyset = A$$

Idempotent laws

$$A \cup A = A$$
$$A \cap A = A$$

Double complement

$$\overline{\overline{A}} = A$$

Absorption laws

$$A \cup (A \cap B) = A$$
$$A \cap (A \cup B) = A$$

Complement laws

$$A \cup \overline{A} = U$$
$$A \cap \overline{A} = \emptyset$$

Abstract algebraic systyem:

Boolean Algebra.

- Duality in logic

  swap: $\wedge \Leftrightarrow \vee$, $T \Leftrightarrow F$

- Duality in set theory

  swap: $\cap \Leftrightarrow \cup$, $\mathcal{U} \Leftrightarrow \phi$

Theorem

If two expressions are 'equal', then so are their duals.

(2.7) Quantifiers

so far : Propositional logic
(0th order logic)

now : Predicate logic
Quantifier logic
1st order logic.

Recall a propositional function (also predicate, open sentence) $P(x)$ is a sentence that becomes a statement when a value for x is specified.

Ex. $P(x) = (x < 7)$

$P(2) = (2 < 7)$ T

$P(10) = (10 < 7)$ F

Ex. $P(x) =$ 'x is a sunny day'

$P(\text{today}) =$ 'today is a sunny day'

Ex. $Q(x, y) = (x + y = 10)$

$Q(3, 7) = (3 + 7 = 10)$ T

$Q(3, 8) = (3 + 8 = 10)$ F

The universe of discourse of a propositional function is the set of values that can be substituted for its variable.

notation: $\mathcal{U}$

Another way to obtain statements from Propositional functions.

Quantification

- universal quantifier: $\forall$ (for all)
- Existential quantifier: $\exists$ (there exists, for some)

Defn

- $\forall x \in \mathcal{U} : P(x)$

  $\forall x \; P(x)$ (if $\mathcal{U}$ understood)

'for all $x \in \mathcal{U}$, $P(x)$ is true'

- $\exists x \in \mathcal{U} : P(x)$

  $\exists x\, P(x)$ (if $\mathcal{U}$ understood)

  'there exists an $x \in \mathcal{U}$ such that $P(x)$ is true'

  'there exists at least one $x \in \mathcal{U}$ ....'

  'for some $x \in \mathcal{U}$ ...'

<u>Ex.</u> $P(x) = (x^2 \geq 0)$ $\quad \mathcal{U} = \mathbb{R}$

$\forall x\, P(x) =$ 'the square of any real number is non-negative'

Ex. $Q(x) = (x < 50) \quad U = \mathbb{R}$

$\forall x\, Q(x)$ = 'every real number is less than 50'.

$\exists x\, Q(x)$ = 'some real number is less than 50'.

Ex $R(x) = (x^2 < 0) \quad U = \mathbb{R}$

$\exists x\, R(x)$ = 'some real number has a negative square'.

Ex $R(x) = (x^2 < 0) \quad U = \mathbb{C}$

$\exists x\, R(x)$ = 'there exists a complex number whose square is negative.'

<u>Ex.</u> $P(x,y) = (y^2 = x)$ $\quad U_x = U_y = \mathbb{R}$.

$\forall x \, \exists y \, P(x,y)$ = 'every real number has a real square root'

<u>Ex.</u> $R(x,y,z) = (x \cdot y = z)$

$U_x = U_y = U_z = \mathbb{Q}$

$\forall x \, \forall y \, \exists z : R(x,y,z)$

'the product of any two rational numbers is a rational number'

<u>Ex.</u> 'every real number has a real cube root'.

$$\forall x \in \mathbb{R}\ \exists y \in \mathbb{R} : y^3 = x$$

<u>notice</u>: the order of Quantifier
does 'n critical:

$$\exists y \in \mathbb{R}\ \forall x \in \mathbb{R} : y^3 = x$$

says
'there exists a real number that is the cube root of every real number.'