- all remote through week 4
  mostly does not affect us
- hw due tonight
- quiz 1 on Thursday

Ex.

Suppose $J \neq \phi$, and $J \subseteq I$. Is it true that

$$(*) \qquad \bigcup_{\alpha \in J} A_\alpha \subseteq \bigcup_{\alpha \in I} A_\alpha \quad ?$$

First concrete example

$J = \{1, 2\}$, $I = \{1, 2, 3\}$

$$\bigcup_{\alpha \in J} A_\alpha = A_1 \cup A_2$$

$$\bigcup_{\alpha \in I} A_\alpha = A_1 \cup A_2 \cup A_3$$

so we're asking

$$A_1 \cup A_2 \subseteq A_1 \cup A_2 \cup A_3 \quad ? \quad \underline{\underline{\text{true}}}$$

<u>Proof of (*)</u>

Suppose $x \in \bigcup_{\alpha \in J} A_\alpha$. Then for some $\beta \in J$, $x \in A_\beta$. But since $J \subseteq I$, we have $\beta \in I$.

Since $x \in A_\beta$, we have

$$x \in \bigcup_{\alpha \in I} A_\alpha .$$

We showed that if $x \in \bigcup_{\alpha \in J} A_\alpha$,

then $x \in \bigcup_{\alpha \in I} A_\alpha$, and therefore

$$\bigcup_{\alpha \in J} A_\alpha \subseteq \bigcup_{\alpha \in I} A_\alpha$$

$\blacksquare$

<u>Question</u>: How do we interpret

$$\bigcup_{\alpha \in I} A_\alpha \quad \text{and} \quad \bigcap_{\alpha \in I} A_\alpha$$

when $I = \phi$?

Answer:

$$\bigcup_{\alpha \in \phi} A_\alpha = \phi$$

identity element with respect to union: $S \cup \phi = S$

The intersection over $I = \phi$ makes sense only relative to a 'universe' $U = \{\text{all objects}\}$, i.e. $A_\alpha \subseteq U$ for all α.

$$\bigcap_{\alpha \in \phi} A_\alpha = U$$

identity with respect to intersection $S \cap U = S$ (as long as $S \subseteq U$).

Thus $(\ast\ast)$ is true even in case $J = \phi$.

(1.9) Number systems

we know

$$\underbrace{\mathbb{N}}_{} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

has special property:

well ordering Property (W.O.P.)

Any non-empty subset of $\mathbb{N}$ contains a least element.

$$\boxed{\text{i.e., if } A \subseteq \mathbb{N} \text{ and } A \neq \phi, \text{ then there exists } x_0 \in A \text{ such that } x_0 \leq x \text{ for all } x \in A.}$$

This is such a basic fact of $\mathbb{N}$ that it's not a theorem, it's an <u>Axiom</u>.

<u>Number Theory</u>: study of $\mathbb{N}$ and $\mathbb{Z}$.

<u>Theorem</u> (Division Algorithm)

Given $a, b \in \mathbb{Z}$ with $b > 0$ ($a \geq 0$), there exist unique $q, r \in \mathbb{Z}$ s.t.

$$a = q \cdot b + r \quad \text{and} \quad 0 \leq r < b$$

dividend   divisor

quotient   remainder

$$\begin{array}{l} a \longrightarrow \\ b \longrightarrow \end{array} \boxed{\begin{array}{c}\text{int}\\ \text{div}\end{array}} \begin{array}{l} \longrightarrow q = a\ \underline{\text{div}}\ b = a/b \\ \longrightarrow r = a\ \underline{\text{mod}}\ b = \underbrace{a \% b}_{\text{most comp. languages}} \end{array}$$

Two things to prove.

(1) <u>existence</u> of q and r.

(2) <u>uniqueness</u> of q and r.

<u>Proof of (1)</u>

Let

$$A = \left\{ a - xb \;\middle|\; x \in \mathbb{Z} \text{ and } 0 \le a - xb \le a \right\}$$

observe that $a \in A$, so $A \neq \emptyset$. Also

$$A \subseteq \{0, 1, 2, \ldots, a\} \subseteq \underbrace{\{0\} \cup \mathbb{N}}_{\text{non-negative ints}}$$

also satisfy W.O.P.

By the W.O.P, A contains a least element, call it r.

Thus

$$r = a - qb \text{ for some } q \in \mathbb{Z}.$$

and

$$r \geq 0$$

we must show $r < b$. observe that if $r \geq b$, we would have

$$0 \leq r - b = (a - qb) - b = a - (q+1)b < a - qb$$

Then $a-(q+1)b \in A$, $a-(q+1)b \geq 0$ but $a-(q+1)b < a - qb$. But $a-qb$ was supposed to be the smallest such number.

we conclude that $r \geq b$ must be false, so $r < b$. Thus

$$a = qb + r \quad \text{and} \quad 0 \leq r < b.$$

This proves existence of q and r.

■

<u>Exercise</u>

Try to prove uniqueness. i.e.

suppose

$$a = q_1 b + r_1 \quad \text{and} \quad 0 \leq r_1 < b$$

and

$$a = q_2 b + r_2, \quad \text{and} \quad 0 \leq r_2 < b$$

Then show $q_1 = q_2$ and $r_1 = r_2$.

(1.10) Russel's Paradox

Question:

Can there be a set that has itself as a member.

$$A = \{A\} = \{\{A\}\} = \{\{\{A\}\}\}$$

$$A = \{\{\{\cdots\}\}\}$$

Our naieve definition of set has no obstacle to this. In fact we could also form

$$\mathcal{S} = \{x \mid x \text{ is a set}\}$$

i.e. the set of all sets.

evidently $\mathcal{S} \in \mathcal{S}$. on the other hand $\phi \notin \phi$, since ϕ has no members.

Define

$$\mathcal{R} = \{A \in \mathcal{S} \mid A \notin A\}$$

obviously either $\mathcal{R} \in \mathcal{R}$ or $\mathcal{R} \notin \mathcal{R}$.

- If $\mathcal{R} \in \mathcal{R}$, then $\mathcal{R}$ must satisfy the defining property of $\mathcal{R}$: $\mathcal{R} \notin \mathcal{R}$.

- If $R \notin R$, then R does satisfy the defining property of R, hence $R \in R$.

We've shown $R \in R$ iff $R \notin R$.
This is the paradox.

Chapter 2: Logic

(2.1)

Defn a statement (or proposition) is a sentence that can be assigned a truth value: true or false. (at least in principle)

<u>Statements:</u>

- 'it is raining today'
- $2+3=5$
- $2+3=7$

<u>not statements:</u>

- 'hello'
- $x+7=2$
- $a+b=c$

$\left.\begin{matrix} x+7=2 \\ a+b=c \end{matrix}\right\}$ called <u>open sentences</u> or <u>propositional functions</u> or <u>predicates</u>.

- we use letters: p, q, r, s ..... to stand for statements.
- we use functional notation: $P(x)$, $Q(x)$, ... to stand for propositional functions.