CSE 16 8-7-24

SETS: open Th 8/8

close T 8/13

QUIZ 5: Th 8/8 : 6.1, 6.2, 6.3, 6.4

Final : Th 8/15

6.5 Generalized Permutations & Combinations

Recall

- K-Permutations from an n-set (without repetition):

$$P(n,k) = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!} \quad (0 \le k \le n)$$

- k-Permutations from an n-set with repetition. (strings)

$$= n^k$$

- k-combinations from an n-set (without repetition) (i.e. subsets)

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

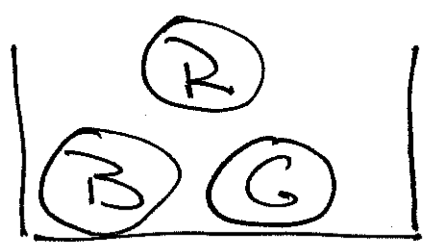
We need generalized combinations, i.e. with repetition.

Defn

A <u>k-combination</u> of a set S <u>with repetition</u> is an unordered arrangement its elements where repetition is allowed. This is also called a <u>k-multiset</u> based on S

Ex.

An urn contains 3 Balls: Red, Blue, Green



Draw 4 balls with replacement. How many color combinations are Possible?

RRRR, BBBB, GGGG

RRRB, RRRG,

BBBR, BBBG,

GGGR, GGGB

RRBB, RRGG, BBGG

RRBG, BBRG, GGRB

$\boxed{15}$ combinations

We've counted the 4-combinations from $\{R, B, G\}$ with repetition.

Also called a 4-multiset based on $\{R, B, G\}$

Notice each combination corresponds to a bit string {*, |}.

RRBG ⟷ **|*|*

RBBB ⟷ *|***|

BGGG ⟷ |*|***

RRGG ⟷ **||**

RRRB ⟷ ***|*|

Each bit string is of len. 6 and contains 4 *'s and 2 |'s. The 2 bars | divide the string into 3 cells representing colors RBG. The number of stars *'s in each cell gives the # of times that color is drawn.

BBGG $\longleftrightarrow$ |**|**

GGGG $\longleftrightarrow$ ||****

so

color combinations

= # bit strs of len 6 with 4 *'s (or 2 |'s)

$$= \binom{6}{4} = \binom{6}{2} = \frac{6!}{4!2!} = \boxed{15}$$

Theorem

Let $n, k \in \mathbb{N}$. The number of k-combinations from an n-set with repetition is

$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$$

Proof

Let $S = \{1, 2, \ldots, n\}$. each k-comb from S with rep. corresponds to a string on $\{*, |\}$ with

$$\begin{cases} k \text{ stars } * \\ n-1 \text{ bars } | \end{cases}$$

The n-1 bars partition the string into n cells representing the n element of S.

Cell ① | Cell ② | .... | Cell(n)

the # of $*$'s in each cell represents the # of times the corresponding element is selected.

The mapping

$\{$ k-comb from S with rep. $\} \leftrightarrow \{$ bit strings of len $n+k-1$ with k stars $\}$

is a bijection. (exercise Prove this)

Thus

\# k-comb from S with rep.

= \# bit strs. of len $n+k-1$ with k *s.

$$= \binom{n+k-1}{k} = \binom{n+k-1}{n-1}. \qquad \blacksquare$$

Ex. $n=2$, $k=3$

Same Prob. but 2 colors, 3 draws.

RRR $\rightarrow$ ***|

RRB $\rightarrow$ **|*

RBB $\rightarrow$ *|**

BBB $\rightarrow$ |***

$$\binom{4}{3} = \binom{4}{1} = \boxed{4}$$

Ex.

Determine the sum

$$\sum_{k=1}^{10} \sum_{j=1}^{k} \sum_{i=1}^{j} 1$$

Each term corresponds to a unique triple (i, j, k) satisfying

$$1 \le i \le j \le k \le 10$$

These are the 3-combinations from $\{1, 2, \ldots, 10\}$ with repetition.

$$\therefore \text{Sum} = \binom{10+3-1}{3} = \binom{12}{3} = \boxed{220}$$

By contrast

Ex. find

$$\sum_{k=3}^{10} \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} 1$$

Each term corresponds to a unique triple (i, j, k) with

$$1 \le i < j < k \le 10.$$

These are 3-comb. from $\{1, \dots, 10\}$ (without repetition).

$$\therefore \text{Sum} = \binom{10}{3} = \boxed{120}$$

Ex.

How many solutions are there to

$$x_1 + x_2 + x_3 = 11$$

where $x_i \in \mathbb{N}$ $(1 \le i \le 3)$?

Each solution corresponds to an 11-combination from $\{1, 2, 3\}$ with repetition.

$$x_1 = \# \text{ of } 1\text{'s}$$

$$x_2 = \# \text{ of } 2\text{'s}$$

$$x_3 = \# \text{ of } 3\text{'s}$$

so

$$\# \text{solns} = \binom{3+11-1}{11} = \binom{13}{11} = \boxed{78}$$

Summary

The # of arrangements of k elements from an n-set is

Ordered? \ repetition?	no	yes
no (combin.)	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$	$\binom{n+k-1}{k}$
yes (Permut.)	$P(n,k) = \frac{n!}{(n-k)!}$	n^k

Permutations with some objects indistinguishable

<u>EX.</u>
How many anagrams of 'EVERGREEN' are there?

$$\left.\begin{array}{l} EEEE \\ G \\ N \\ RR \\ V \end{array}\right\} \quad \text{let } m = \#\text{ anagrams}$$

Temporarily mark E's & R's to make them distinguishable

$$E_1, E_2, E_3, E_4, \quad R_1, R_2$$

The # of Permutations of these 9 distinguishable objects is $9!$

We count these permutations in another way.

- choose an anagram

  $\# \text{ways} = m$

- choose a permutation $\{E_1, E_2, E_3, E_4\}$

  $\# \text{ways} = 4!$

- choose a permutation $\{R_1, R_2\}$

  $\# \text{ways} = 2!$

By the product rule

$$9! = m \cdot 4! \cdot 2!$$

$$\therefore \; m = \frac{9!}{4!\,2!} = \frac{9!}{4!\,2!\,1!\,1!\,1!}$$

$$= \boxed{7{,}560}$$

Theorem

The # of permutations of n objects when there are n_i indistinguishable objects of type i $(1 \le i \le k)$ and $n_1 + n_2 + \cdots + n_k = n$ is

$$\frac{n!}{n_1! \cdot n_2! \cdots n_k!}$$

In last example: $k=5$, $n_1=4$ (E's), $n_2=n_3=n_4=1$ (G, N, V), $n_4=2$ (R's).

Exercise

Prove this.

Ex. find the # of ternary strings (alphabet $=\{0,1,2\}$) of length 20 with exactly 3 0's, 12 1's (and 5 2's).

These strings are anagrams of

$$'\underbrace{000}_{3}\underbrace{111111111111}_{12}\underbrace{22222}_{5}'$$

By last thm.

$$\text{ans} = \frac{20!}{3!\,12!\,5!} = \boxed{7{,}054{,}320}$$

Recall: We already did this

$$\text{ans} = \binom{20}{3} \cdot \binom{17}{12} = \frac{20!}{3!\,\cancel{17!}} \cdot \frac{\cancel{17!}}{12!\,5!}$$

$$= \boxed{7{,}054{,}320}$$

$\binom{20}{3}$: choose 0's

$\binom{17}{12}$: choose 1's

Defn

Let $n = n_1 + n_2 + \cdots + n_k$ where $n_i \in \mathbb{N}$ $(1 \le i \le k)$. The <u>multinomial coefficient</u>

$$\binom{n}{n_1, n_2, \ldots, n_k}$$

is the # of permutations of the n-element multiset

$$\{x_1^{n_1}, x_2^{n_2}, \ldots, x_k^{n_k}\},$$

i.e. # of anagrams of

$$\overbrace{x_1 \cdots x_1}^{n_1}\ \overbrace{x_2 \cdots x_2}^{n_2} \cdots\cdots \overbrace{x_k \cdots x_k}^{n_k}$$

By last thm

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!\, n_2! \ldots n_k!}$$