CSE 16 8-6-24

SETS: opens Th 8/8/24

closes T 8/13/24

Recall: Pascal's identity:

(1) $$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \quad (1 \le k \le n)$$

or

(2) $$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad (1 \le k < n)$$

Proof of (1) (combinatorial)

Let T be a set with $|T| = n+1$.

Let $x \in T$, $S = T - \{x\}$, so $|S| = n$

The task of constructing a k-subset of T (where $1 \le k \le n$) decomposes into 2 subtasks, exactly one of which will be performed.

- include x : # ways $= \binom{n}{k-1}$
  choose a $(k-1)$-subset of S, then add x.
- exclude x : # ways $= \binom{n}{k}$
  choose a k-subset of S

By the sum rule:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}. \qquad \blacksquare$$

<u>Theorem</u>

For all $n \geq 0$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

<u>Proof</u> (combinatorial)

Both sides count the # of subsets of an n-set. $\blacksquare$

The Binomial Theorem

observe

$$(x+y)^0 = 1$$

$$(x+y)^1 = 1\cdot x + 1\cdot y$$

$$(x+y)^2 = 1\cdot x^2 + 2\cdot xy + 1\cdot y^2$$

$$(x+y)^3 = 1\cdot x^3 + 3\cdot x^2 y + 3\cdot xy^2 + 1\cdot y^3$$

$$(x+y)^4 = 1\cdot x^4 + 4\cdot x^3 y + 6x^2y^2 + 4\cdot xy^3 + 1\cdot y^4$$

$$\vdots \qquad\qquad \vdots$$

Each term in $(x+y)^n$ is of the form $x^{n-k}y^k$ (for $0 \le k \le n$), with coeff. from Pascal's Δ.

Theorem (Binomial Thm)

Let $n \in \mathbb{N}$, and $x, y \in \mathbb{R}$. Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

EX. $n=3$

$$(x+y)^3 = \underset{1}{(x+y)}\underset{2}{(x+y)}\underset{3}{(x+y)}$$

choose which factors contribute y!

set	bit string	monomial	
$\emptyset$	000	$xxx = x^3$	x^3
$\{3\}$	001	$xxy = x^2y$	
$\{2\}$	010	$xyx = x^2y$	$3x^2y$
$\{2,3\}$	011	$xyy = xy^2$	
$\{1\}$	100	$yxx = x^2y$	$3xy^2$
$\{1,3\}$	101	$yxy = xy^2$	
$\{1,2\}$	110	$yyx = xy^2$	
$\{1,2,3\}$	111	$yyy = y^3$	y^3

After collecting like terms

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

Proof (combinatorial)

when

$$* \quad (x+y)^n = \overset{①}{(x+y)}\overset{②}{(x+y)} \cdots \overset{ⓝ}{(x+y)}$$

is expanded, and before any like terms are combined, we have 2^n terms, each of the form

$$x^{n-k}y^k \quad (\text{for some } 0 \le k \le n)$$

Fix a particular such k. To determine the coeff. of $x^{n-k}y^k$ we must count the # of ways this term is obtained in the expansion

To get $x^{n-k} y^k$ we select y from k factors in $*$, and x from the remaining $n-k$ factors

To choose which of the k factors will contribute y is to choose a k-subset of $\{1, 2, \ldots, n\}$.

$$\# \text{ways} = \binom{n}{k}$$

Hence $x^{n-k}y^k$ occurs $\binom{n}{k}$ times in the expansion, and after combining like terms, the coeff. of $x^{n-k}y^k$ is $\binom{n}{k}$.

Thus

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k .$$

$\blacksquare$

<u>Corollary</u> $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

<u>Proof</u> Let $x=y=1$ in the Binomial Thm.

$\blacksquare$

<u>corollary</u> $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

<u>Proof</u> let $x=1$, $y=-1$ in the Binomial Thm. ∎

The Algebraic Proof of B. Thm. uses Pascal's identity:

$$\boxed{\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}}$$ for $1 \le k \le n$

Proof (algebraic)

We show

$$\forall n \geq 0: \quad \boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k}$$

$P(n)$

I. $P(0)$ says $(x+y)^0 = \binom{0}{0} x^0 y^0$,
i.e. $1 = 1$, which is true.

IIa. $\forall n \geq 0: P(n) \rightarrow P(n+1)$

Let $n \geq 0$. Assume

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

we must show

$$(x+y)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^{(n+1)-k} y^{k}.$$

observe

$$(x+y)^{n+1} = (x+y)(x+y)^{n}$$

$$= (x+y)\left(\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k}\right) \left\{ \begin{array}{l} \text{by the} \\ \text{ind. hyp.} \end{array} \right.$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k+1} y^{k} + \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k+1}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k+1} y^{k} + \sum_{k=1}^{n+1} \binom{n}{k-1} x^{n-(k-1)} y^{(k-1)+1}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n+1-k} y^{k} + \sum_{k=1}^{n+1} \binom{n}{k-1} x^{n+1-k} y^{k}$$

$$= \binom{n}{0} x^{n+1} y^{0} + \left[\sum_{k=1}^{n} \binom{n}{k} x^{n+1-k} y^{k} + \sum_{k=1}^{n} \binom{n}{k-1} x^{n+1-k} y^{k} \right]$$

$$+ \binom{n}{n} x^{0} y^{n+1}$$

$$= \binom{n+1}{0} x^{n+1} y^{0} + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1} \right] x^{n+1-k} y^{k}$$

$$\binom{n+1}{n+1} x^{0} y^{n+1}$$

$$= \binom{n+1}{0} x^{n+1} y^{0} + \sum_{k=1}^{n} \binom{n+1}{k} x^{n+1-k} y^{k} + \binom{n+1}{n+1} x^{0} y^{n+1}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} x^{(n+1)-k} y^{k}. \qquad \blacksquare$$

Read

- VanderMonde identity
- Corollaries

Ex.
find the coefficients of x^8 and x^{11} in expansion of $(x^2+3)^{20}$.

Solution
We have

$$(x^2+3)^{20} = \sum_{k=0}^{20} \binom{20}{k} (x^2)^{20-k} \cdot 3^k$$

$$= \sum_{k=0}^{20} 3^k \binom{20}{k} x^{40-2k}$$

- $40-2k=8 \to 32=2k \to \boxed{k=16}$

$$(\text{coef. of } x^8) = \boxed{3^{16}\cdot\binom{20}{16}}$$

- $40-2k=11 \to 29=2k$ ✗.

$$(\text{coef. of } x^{11}) = 0$$

<u>Ex.</u> Let $a, b, n \in \mathbb{N}$ with

$$0 \le a+b \le n$$

Determine the # of ternary strings (alphabet $=\{0,1,2\}$) of length n with exactly a 0's and b 1's.

count strings in 2 ways.

$$\#\text{strings} = \binom{n}{a}\binom{n-a}{b}$$

choose Pos. of 0's

choose Pos. of 1's

$$\#\text{strings} = \binom{n}{b}\binom{n-b}{a}$$

choose 1's

choose 0's.

Thus

$$\binom{n}{a}\binom{n-a}{b} = \binom{n}{b}\binom{n-b}{a}$$

Exercise

show

$$\binom{n}{a}\binom{n-a}{b}\binom{n-a-b}{c} = \binom{n}{b}\binom{n-b}{a}\binom{n-a-b}{c}$$

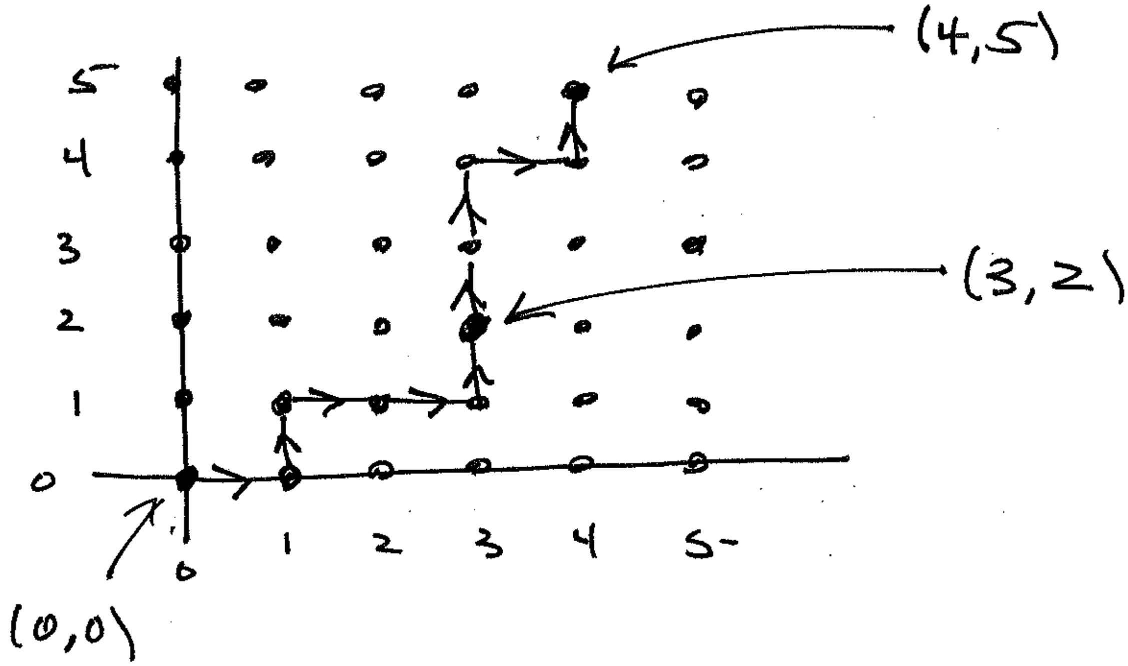
$$= \begin{bmatrix} \text{any permutation} \\ \text{of } \{a, b, c\} \end{bmatrix}$$

where $0 \le a+b+c \le n$. Hint

count 4-ary strings: alpha $= \{0,1,2,3\}$

Ex. Lattice Paths

Count # of lattice Paths in $\mathbb{Z} \times \mathbb{Z}$ from $(0,0)$ to $(4,5)$



consisting of single unit steps

UP ↑ or right →

Each such Path is encoded by a bit string (alphabet $= \{U, R\}$) with exactly 4 R's and 5 U's.

so

$$\text{ans} = \binom{9}{4} = \binom{9}{5} = \boxed{126}$$

How many of these Paths Pass through (3,2) ?

By the Product rule

$$\text{ans} = \binom{5}{3} \cdot \binom{4}{1} = 10 \cdot 4 = \boxed{40}$$