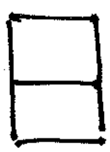
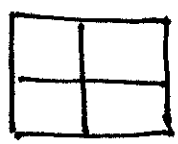
CSE 16 8-14-24

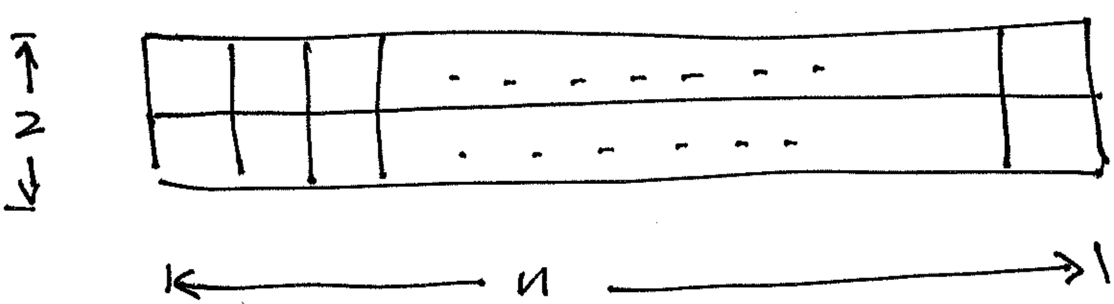
• Final: tomorrow Th 8/15

11:15 am - 1:00 pm

Ex: Given two types of tiles

2x1  2x2 

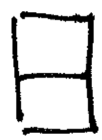
walkway

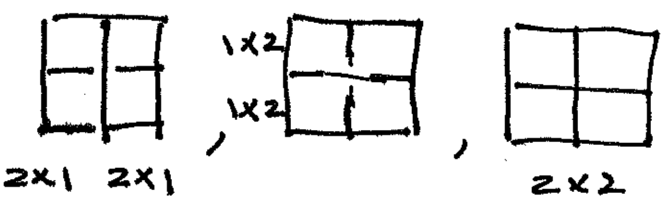


How many tilings are there of the walkway.

Let A_n = # of tilings.

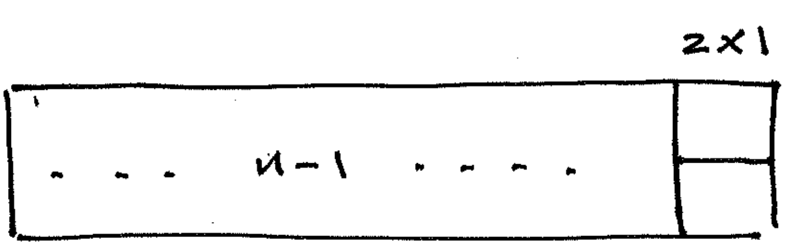
$A_0 = 1$ (empty tiling)

$A_1 = 1$ ()

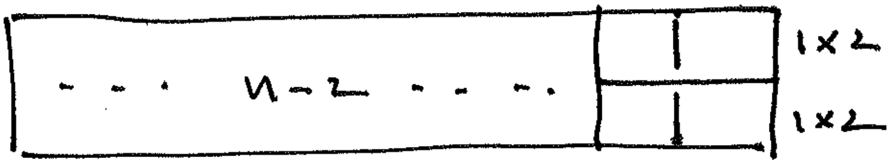
$A_2 = 3$ ()

We can construct a tiling by performing exactly one of the following (mutually exclusive & exhaustive) subtasks

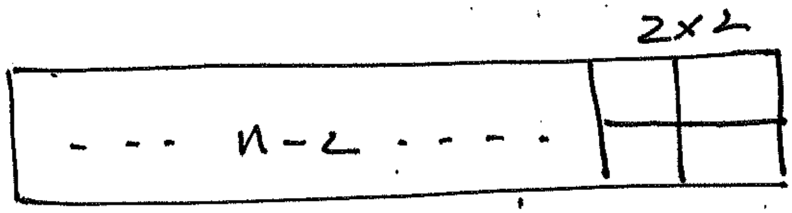
 # ways

(1) 

A_{n-1}

(2) 

A_{n-2}

(3) 

A_{n-2}

Thus, by the sum rule

$$\begin{cases} A_n = A_{n-1} + 2A_{n-2} \\ A_0 = A_1 = 1 \end{cases}$$

$A_2 = 1 + 2 \cdot 1 = 3$

$A_3 = 3 + 2 \cdot 1 = 5$

$A_4 = 5 + 2 \cdot 3 = 11$

$A_5 = 11 + 2 \cdot 5 = 21$

$A_6 = 21 + 2 \cdot 11 = 43$

$$A_7 = 43 + 2 \cdot 21 = 85$$

$$A_8 = 85 + 2 \cdot 43 = 171$$

$$A_9 = 171 + 2 \cdot 85 = 341$$

$$A_{10} = 341 + 2 \cdot 171 = 683$$

$$\vdots$$

- find the Prob. that a tiling of length 10 ends with $\boxplus$ 2x2.

$$\#(\text{tilings of len. } 10) = 683$$

$$\#(\text{tilings of len. 10 ending } \boxplus) = 171$$

so $\text{Prob.} = \dfrac{171}{683} = \boxed{.2504}$.

- check solution

$$\boxed{A_n = \frac{1}{3}\left(2^{n+1} + (-1)^n\right)}$$

init. cond. ✓

$$A_0 = \frac{1}{3}(2+1) = 1 \quad ✓$$

$$A_1 = \frac{1}{3}(4+(-1)) = 1 \quad ✓$$

recurrence : $A_n = A_{n-1} + 2A_{n-2}$ ✓

$$RHS = A_{n-1} + 2A_{n-2}$$

$$= \frac{1}{3}\left(2^n + (-1)^{n-1}\right) + 2\cdot\frac{1}{3}\left(2^{n-1} + (-1)^{n-2}\right)$$

$$= \frac{1}{3}\left(2^n + 2^n + \cancel{(-1)^{n-1}} + \cancel{(-1)^{n-2}} + (-1)^{n-2}\right)$$

$$= \frac{1}{3}\left(2\cdot 2^n + (-1)^n\right)$$

$$= \frac{1}{3}\left(2^{n+1} + (-1)^n\right) = A_n = LHS$$

Review Problems 8.1

#9 – 14, 24 – 27

Ex. find a recurrence for the # of bit strings of len. n that contain '01'

Let B_n = # of such bit strings

note

$B_0 = 0$ $\quad \phi$

$B_1 = 0$ $\quad \phi$

$B_2 = 1$ $\quad \{01\}$

$B_3 = 4$ $\quad \{010, 011, 001, 101\}$

Subtasks

			# ways
(1)	$\underbrace{x \cdots x}_{n-1}\, 0$	:	B_{n-1}
(2)	$\underbrace{x \cdots\cdots x}_{n-2}^{\text{any}}\, 01$	:	2^{n-2}
(3)	$\underbrace{x \cdots x}_{n-3}^{\text{any}}\, 011$	:	2^{n-3}
(4)	$\underbrace{x \cdots x}_{n-4}^{\text{any}}\, 0111$	:	2^{n-4}
$\vdots$			
(n)	$\underbrace{01 1 \cdots\cdots 111}_{n}$	:	$2^{n-n} = 1$

By the sum rule

$$B_n = B_{n-1} + \sum_{k=0}^{n-2} 2^k$$

i.e.

$$B_n = B_{n-1} + \left(\frac{2^{n-1} - 1}{2 - 1}\right)$$

$$\therefore \boxed{B_n = B_{n-1} + 2^{n-1} - 1}$$

init. cond. $\boxed{B_0 = 0}$

$$\therefore B_1 = 0 + 2^0 - 1 = 0 \checkmark$$

$$B_2 = 0 + 2^1 - 1 = 1 \checkmark$$

$$B_3 = 1 + 2^2 - 1 = 4 \checkmark$$

$$B_4 = 4 + 2^3 - 1 = 11$$

$$\vdots$$

<u>check solution</u>: $\boxed{B_n = 2^n - n - 1}$

Another way:

let A_n = # of bit strings of length n that do not contain substring '01'

Observe. $A_n + B_n = 2^n$, hence

$B_n = 2^n - A_n$.

note: a bit string does not contain '01' iff all 1s occur to the left of all 0s.

i.e.

$$\left.\begin{array}{cccccccc}
0 & 0 & 0 & \cdots & \cdots & & & 0 \\
1 & 0 & 0 & \cdots & \cdots & & & 0 \\
1 & 1 & 0 & \cdots & \cdots & & & 0 \\
1 & 1 & 1 & 0 & \cdots & & & 0 \\
\vdots & & & & & & & \\
1 & 1 & 1 & 1 & \cdots & & 1 & 0 \\
1 & 1 & 1 & 1 & \cdots & & 1 & 1
\end{array}\right\} A_n = n+1$$

$$\underbrace{\hphantom{0\ 0\ 0\ \cdots\cdots\ 0}}_{n}$$

Thus

$$B_n = 2^n - A_n = 2^n - (n+1)$$

$$\therefore \boxed{B_n = 2^n - n - 1}$$

Ex. (8.1) #12a

A kid climbs stairs by taking

- 1 step
- 2 steps
- 3 steps

let C_n = # ways he can climb a staircase of length n. Find a recurrence for C_n.

Initial terms:

$C_0 = 1$ (stand still)

$C_1 = 1$ (1 step)

$C_2 = 2$ $\{(1,1), (2)\}$

$C_3 = 4$ $\{(1,1,1), (1,2), (2,1), (3)\}$

subtasks: | #ways

• climb $n-1$ steps, (1) :	c_{n-1}
• " $n-2$ " , (2) :	c_{n-2}
• " $n-3$ " , (3) :	c_{n-3}

By the sum rule

$$\begin{cases} c_n = c_{n-1} + c_{n-2} + c_{n-3} \\ c_0 = 1 \\ c_1 = 1 \\ c_2 = 2 \end{cases}$$

so

$$c_3 = 2+1+1 = 4$$

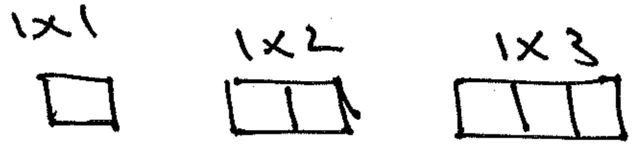
$$c_4 = 4+2+1 = 7$$

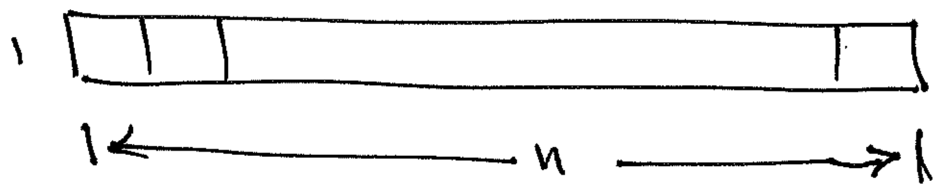
$$c_5 = 7+4+2 = 15$$

$$\vdots$$

<u>note</u>

same problem as tiling question

tiles 

walkway 

how many tilings ?

8.1 # 27

tiles: $\boxed{R}$ $\boxed{B}$ $\boxed{G}$

· no two R adjacent

Let T_n = # of such tilings

$T_0 = 1$ empty

$T_1 = 3$ $\{R, B, G\}$

$T_2 = 8$ $\{RB, RG, BB, BR, BG, GG, GR, GB\}$

substance	# ways

- no RR: $[\; n-1 \;|\; B \;]$ — T_{n-1}
- no RR: $[\; n-1 \;|\; G \;]$ — T_{n-1}
- no RR: $[\; n-2 \;|\; G \;|\; R \;]$ — T_{n-2}
- no RR: $[\; n-2 \;|\; B \;|\; R \;]$ — T_{n-2}

By sum rule:

$$\begin{cases} T_n = 2T_{n-1} + 2T_{n-2} \\ T_0 = 1 \\ T_1 = 3 \end{cases}$$

so $T_2 = 2\cdot 3 + 2 \cdot 1 = 8$ ✓

Ex.

What is the probability that a random int in $\{1, 2, \ldots, 100\}$ is divisible by 7 ?

$$S = \{1, \ldots, 100\} \qquad |S| = 100$$

$$E = \{7, 14, 21, 28, \ldots, 98\}$$

$$|E| = \left\lfloor \frac{100}{7} \right\rfloor = \lfloor 14.2 \rfloor = 14$$

$$P(E) = \frac{14}{100} = \boxed{.14}$$